The multidimensional East model: a multicolour model and a front evolution problem

Ph.D. thesis defense

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13 June 2022

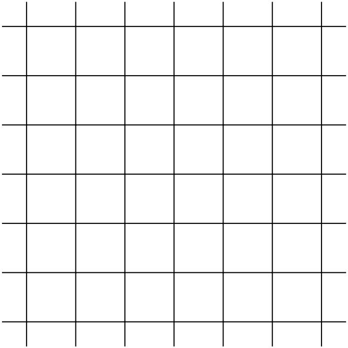
ROMA TRE
UNIVERSITÀ DEGLI STUDI

Plan

- Multidimensional East model
- Front evolution problem
- Multicolour East model (MCEM)

Multidimensional East model

- Markov process on $\mathbb{Z}^d$, parameter $q \in (0, 1)$.
- State space $\{0, 1\}^{\mathbb{Z}^d}$.



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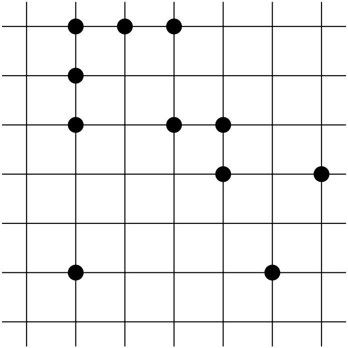

Alternatively:

- $0 =$ vacancy / $\bullet$ / infected.
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Multidimensional East model

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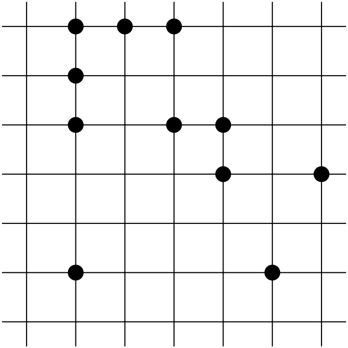

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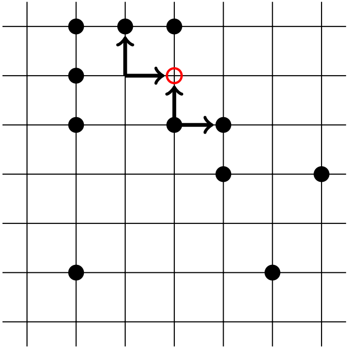

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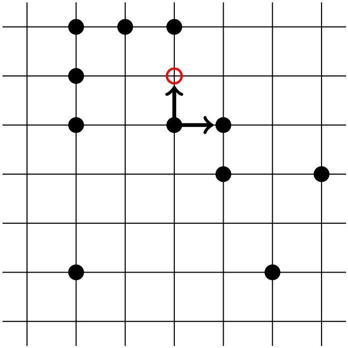

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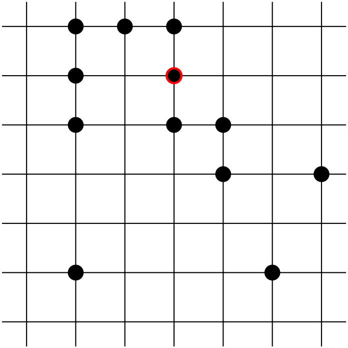
- If legal $\Rightarrow$ sample from $\mu_x = \mathrm{Ber}(p)$, $p = 1 - q$.



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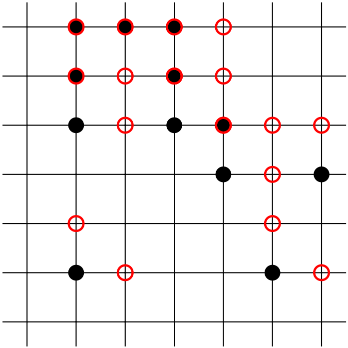
- If legal $\Rightarrow$ sample from $\mu_x = \mathrm{Ber}(p)$, $p = 1 - q$.
- $\mu = \bigotimes_{x \in \mathbb{Z}^d} \mu_x$ reversible.



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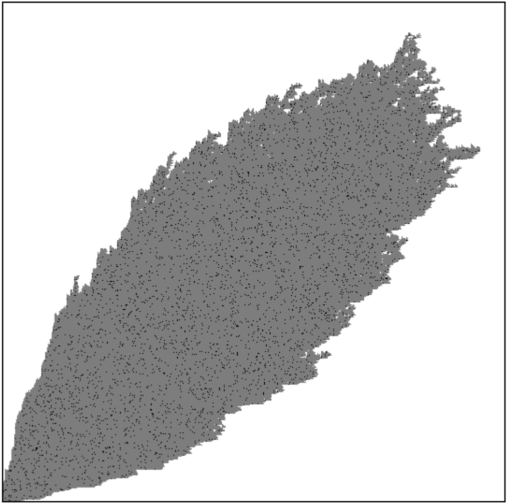
Simulation results

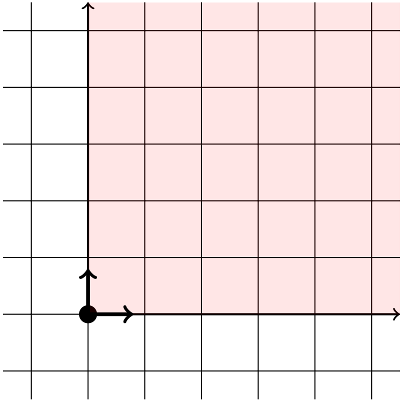


$\bullet$ = previously $\bullet$.

Front evolution problem

- Start with state ω_* with single vacancy at origin.

- $\bullet$ only on first quadrant.



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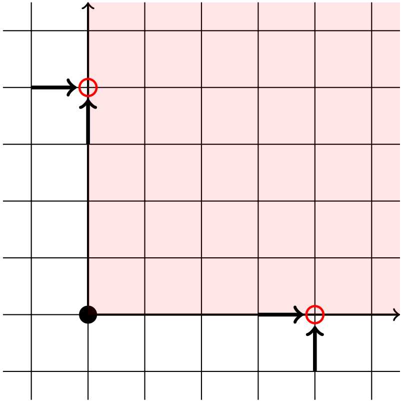
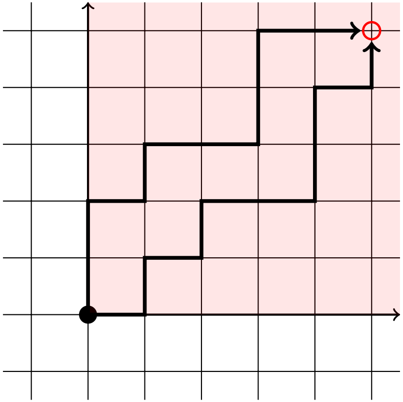
- One-dimensional East along axes.



Front evolution problem

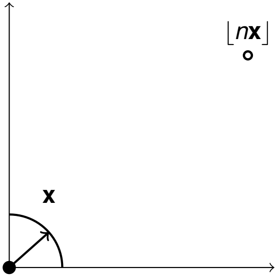
- Start with state ω_* with single vacancy at origin.
- $\bullet$ only on first quadrant.
- One-dimensional East along axes.
- Faster propagation to vertices away from axis.



Question: Is there a front velocity?

$\tau_x =$ infection time of $\lfloor x \rfloor \in \mathbb{Z}_+^d$, $\quad$ $\mathbf{x} =$ unit vector in $\mathbb{R}_+^d$.

$$\frac{1}{v_{\max}(\mathbf{x})} := \liminf_{n\to\infty} \frac{\mathbb{E}_{\omega_*}(\tau_{n\mathbf{x}})}{n}, \qquad \frac{1}{v_{\min}(\mathbf{x})} := \limsup_{n\to\infty} \frac{\mathbb{E}_{\omega_*}(\tau_{n\mathbf{x}})}{n}$$



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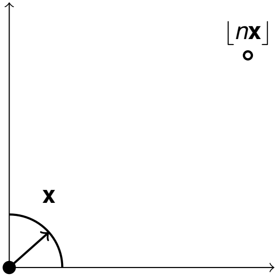


Main problems

Bounds on $v_{\min}(\mathbf{x})$, $v_{\max}(\mathbf{x})$.

Harder: Identify $\mathbf{x}$ for which $v_{\min}(\mathbf{x}) = v_{\max}(\mathbf{x})$.

Question: Is there a front velocity?

Theorem (O. Blondel '13)

In $d = 1$ there exists a $v = v(q)$ such that
$v = v_{\min}(\mathbf{e}_1) = v_{\max}(\mathbf{e}_1)$ *for any q.*

Question: Is there a front velocity?

Theorem (O. Blondel '13)

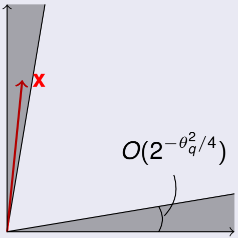
In $d = 1$ there exists a $v = v(q)$ such that
$v = v_{\min}(\mathbf{e}_1) = v_{\max}(\mathbf{e}_1)$ for any q.

No bounds on $v_{\min}(\mathbf{x})$, $v_{\max}(\mathbf{x})$ for $d \geq 2$, $\mathbf{x} \neq \mathbf{e}$.

Small q behaviour of $v_{\max}(\mathbf{x})$, $v_{\min}(\mathbf{x})$

Write $\theta_q = \log_2(1/q)$. By (P. Chleboun, A. Faggionato, F. Martinelli '16) the spectral gap $\gamma_d(q)$ of the East model on $\mathbb{Z}^d$ is $2^{-\frac{\theta_q^2}{2d}(1+o(1))}$.

Theorem (Y.C., F. Martinelli '22)



If $d = 2$, $\mathbf{x}$ as in figure, then

$$\begin{aligned} v_{\max}(\mathbf{x}) &= v_{\min}(\mathbf{x})^{1+o(1)} \\ &= 2^{-\frac{\theta_q^2}{2}(1+o(1))} \\ &= \gamma_1(q)^{1+o(1)}, \quad q \ll 1. \end{aligned}$$

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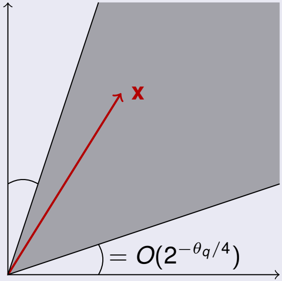

If $d \geq 2$, $\mathbf{x}$ as in figure $\Rightarrow \exists \alpha < 1$ s.t.

$$v_{\min}(\mathbf{x}) \geq 2^{-\frac{\theta_q^2}{2}\alpha} \gg v(\mathbf{e}), \quad q \ll 1.$$

Small q behaviour of $v_{\max}(\mathbf{x})$, $v_{\min}(\mathbf{x})$

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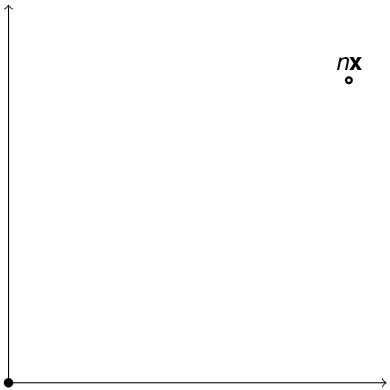
Theorem (Y.C., F. Martinelli '22)

If $d \geq 2$, $\mathbf{x} \in \mathbb{R}_+^d$ *s.t.* $\min_i \mathbf{x}_i > 0$. *Then*

$$v_{\max}(\mathbf{x}) = v_{\min}(\mathbf{x})^{1+o(1)} = 2^{-\frac{\theta_q^2}{2d}(1+o(1))} = \gamma_d^{1+o(1)}(q), \quad q \ll 1.$$

Main ingredients for $v_{\max}(\mathbf{x}) \leq 2^{-\frac{\theta_q^2}{2d}(1+o(1))}$ as $q \to 0$



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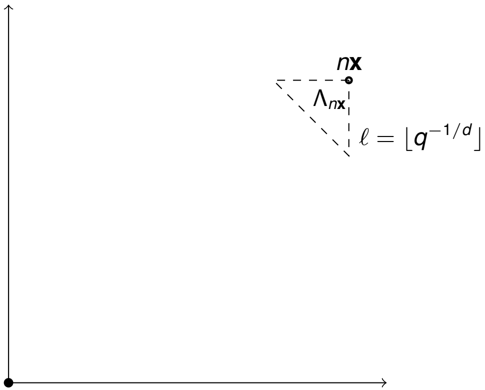

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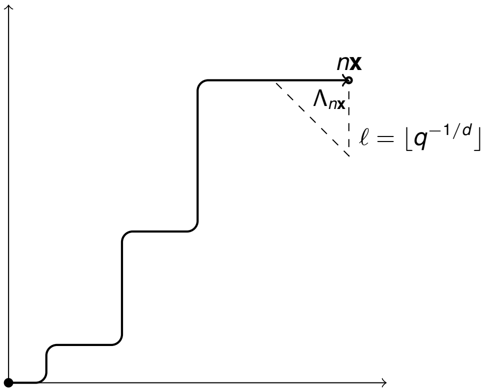

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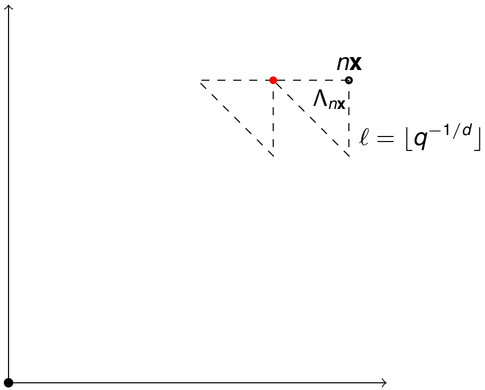

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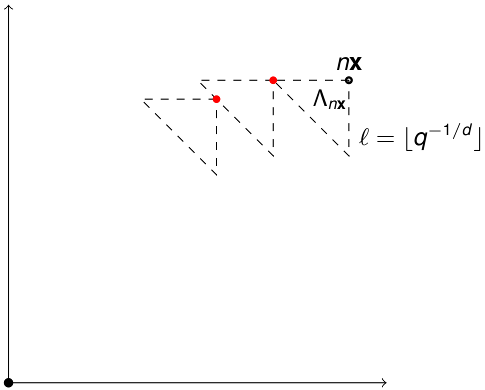

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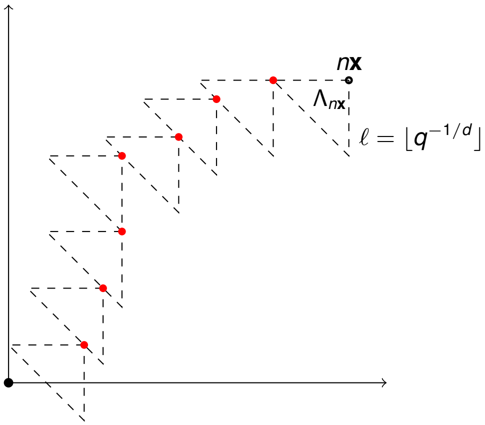
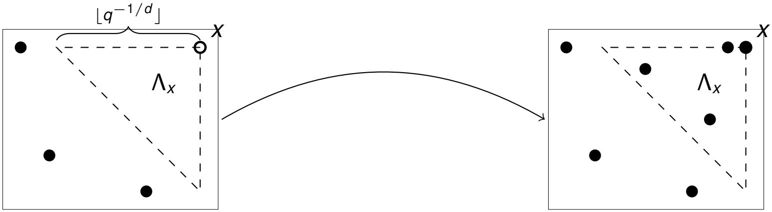


- Show that $\max_{\omega:\text{ no } \bullet \text{ in } \Lambda_x} \mathbb{P}_\omega(\tau_x < t) \to 0$ if $t = o(2^{\frac{\theta_q^2}{2d}})$ as $q \to 0$.

$\max_{\omega:\text{ no }\bullet\text{ in }\Lambda_x} \mathbb{P}_\omega(\tau_x < t) \to 0$ if $t = o(2^{\frac{\theta_q^2}{2d}})$

Going through a bottleneck



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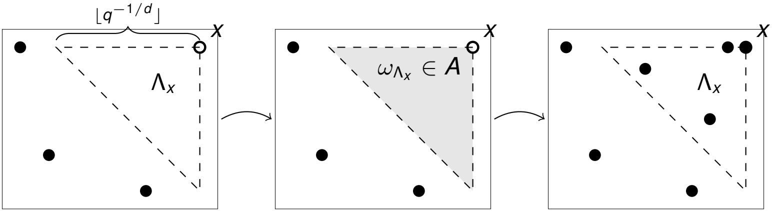

$$\max_{\omega:\ \text{no}\ \bullet\ \text{in}\ \Lambda_x} \mathbb{P}_\omega(\tau_x < t) \le \max_{\omega:\ \text{no}\ \bullet\ \text{in}\ \Lambda_x} \mathbb{P}_\omega(\tau_A < t)$$

- CFM'16: $\exists A \in \Omega_{\Lambda_x}$ with $\mu(A) \le 2^{-\frac{\theta_q^2}{2d}(1+o(1))}$ and $\tau_A < \tau_x$ when starting with no vacancy in Λ_x.

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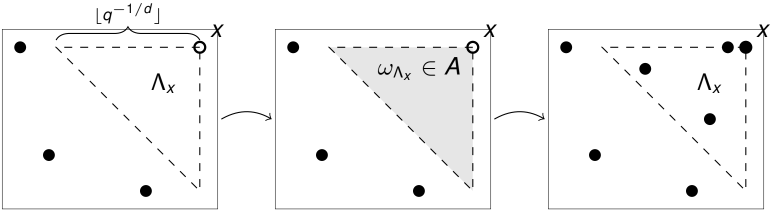

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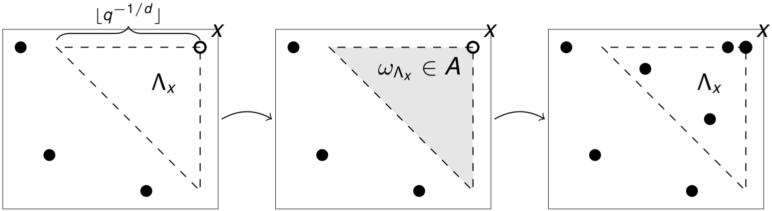

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$$\le O(t) \times 2^{-\frac{\theta_q^2}{2d}(1+o(1))}$$

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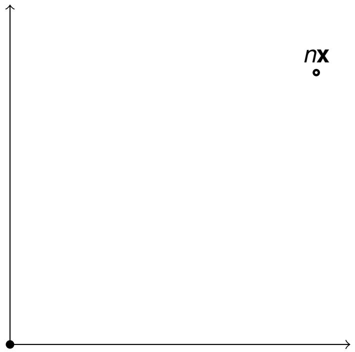
Main ingredients for $v_{\min}(\mathbf{x}) \geq 2^{-\frac{\theta_q^2}{2d}(1+o(1))}$ as $q \to 0$



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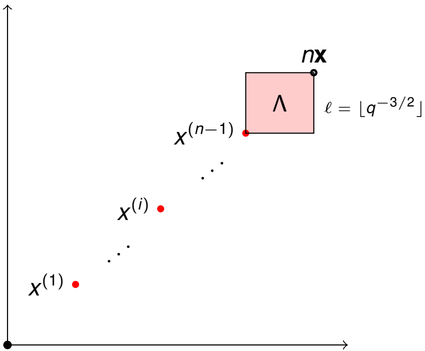

- By SMP show as $q \to 0$:

$$\max_{\omega:\, \omega_{x^{(i)}} = \bullet} \mathbb{P}_\omega(\tau_{x^{(i+1)}} > t) \to 0 \text{ if } t \gg 2^{\frac{\theta_q^2}{2d}}.$$

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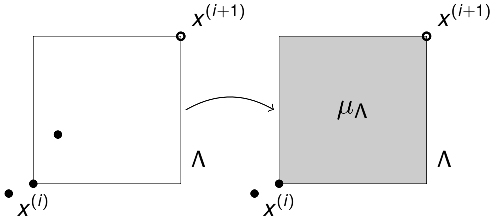

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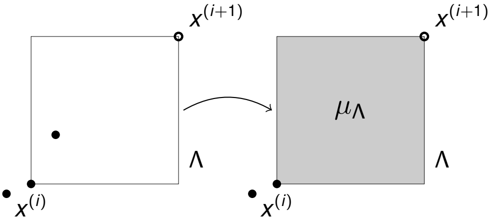

- $\mathbb{P}_\mu(\tau_{x^{(i+1)}} > t) \le e^{-t\lambda_D}$, where λ_D is the smallest λ s.t.

$$-\mathcal{L}_\Lambda f = \lambda f, \qquad f \restriction_{\{\omega:\, \omega_{x^{(i+1)}}=\bullet\}} = 0.$$

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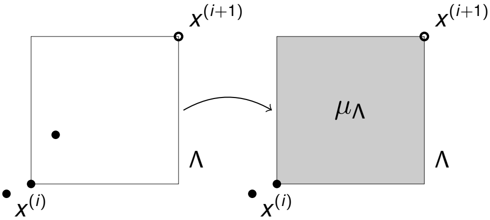

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$$-\mathcal{L}_\Lambda f = \lambda f, \qquad f \restriction_{\{\omega:\, \omega_{x^{(i+1)}}=\bullet\}} = 0.$$

- Bad: $\lambda_D \ge q\gamma_\Lambda(q)$ but $\gamma_\Lambda(q) = \gamma_1^{(1+o(1))}(q) = 2^{-\frac{\theta_q^2}{2}(1+o(1))}$.

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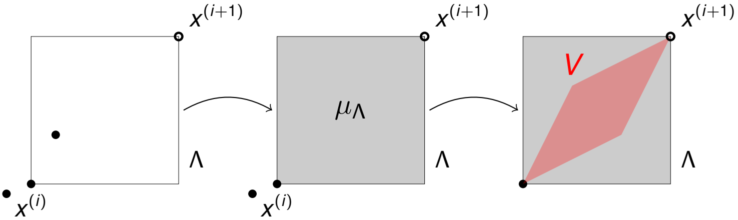

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$$-\mathcal{L}_\Lambda f = \lambda f, \qquad f \restriction_{\{\omega:\, \omega_{x^{(i+1)}}=\bullet\}} = 0.$$

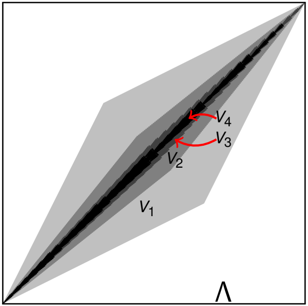
- Better: $\lambda_D \ge q \max\{\gamma_V(q) :\ V \subset \Lambda, V \supset \{0, x^{(i+1)}\}\}$.

$\max_{\omega:\, \omega_{x(i)}=\bullet} \mathbb{P}_\omega(\tau_x > t) \to 0$ if $t \gg 2^{\frac{\theta_q^2}{2d}}$

Proposition (Y.C., F. Martinelli '22)

For $q \to 0$ $\exists V \subset \Lambda$ containing both the lower left and top right corner s.t.

$$\gamma_V(q) \geq 2^{-\frac{\theta_q^2}{2d}(1+o(1))}.$$



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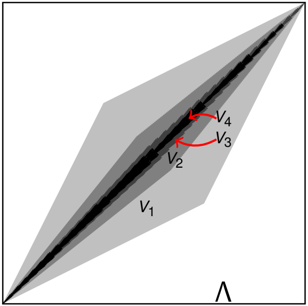
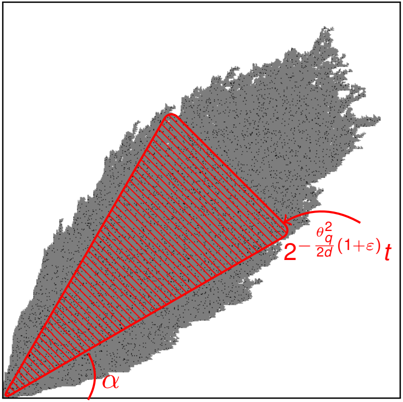


$$\mathbb{P}_\mu(\tau_{x(i+1)} > t) \leq e^{-t2^{-\frac{\theta_q^2}{2d}(1+o(1))}}.$$

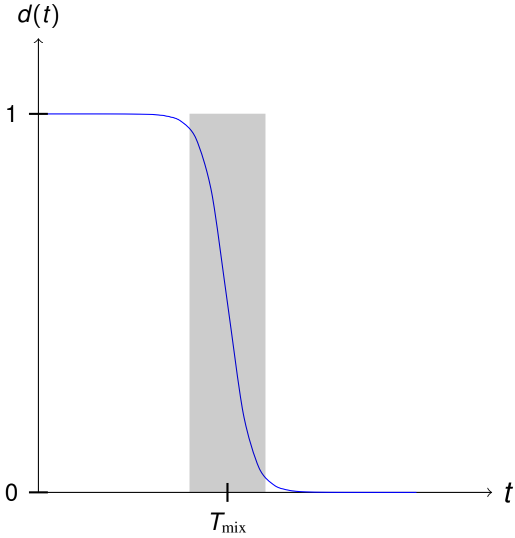
Equilibrium behind front



Theorem (Y.C., F. Martinelli '22)

Vertices in red shape in equilibrium for large t and small q if $\alpha > 0$.

Cutoff



Cutoff

Theorem (S. Ganguly, E. Lubetzky, F. Martinelli '15)

There is a v such that the East process on $\{0, \ldots, n\}$ with parameter $0 < q < 1$ exhibits cutoff at $v^{-1}n$ with window $\sqrt{n}$.

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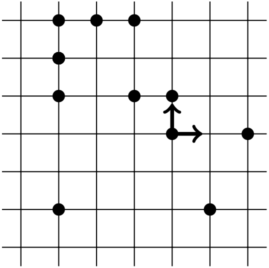
Theorem (Y.C., F. Martinelli '22)

There exists $q_0 > 0$ *such that the East process on* $\{0, \ldots, n\}^d$ *with parameter* $0 < q < q_0$ *exhibits cutoff at* $v^{-1}n$ *with window* $O(n^{2/3})$.

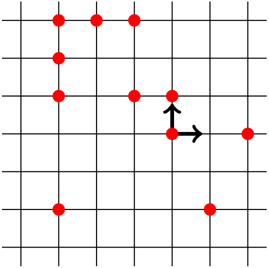
- Because modes away from axes relax much quicker than axes modes!

The multicolour East model on $\mathbb{Z}^2$



- State space $\{\circ, \bullet\}^{\mathbb{Z}^d}$, eq. density q for $\bullet$ and $p = 1 - q$ for $\circ$.

The multicolour East model on $\mathbb{Z}^2$



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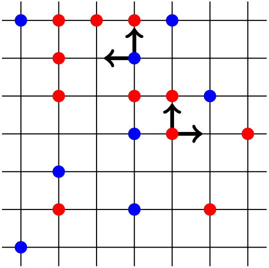

- $\{\circ, \textcolor{red}{\bullet}, \textcolor{blue}{\bullet}\}^{\mathbb{Z}^d}$, eq. density $q_{\textcolor{red}{\bullet}}, q_{\textcolor{blue}{\bullet}}$, $p = 1 - q_{\textcolor{red}{\bullet}} - q_{\textcolor{blue}{\bullet}}$ for $\circ$.
- Only $\circ \leftrightarrow \textcolor{red}{\bullet}$ and $\circ \leftrightarrow \textcolor{blue}{\bullet}$ transitions

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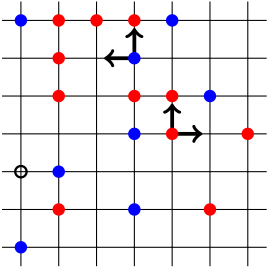

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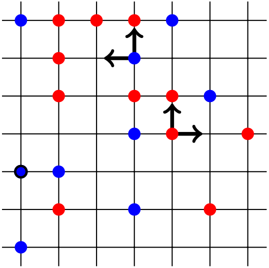

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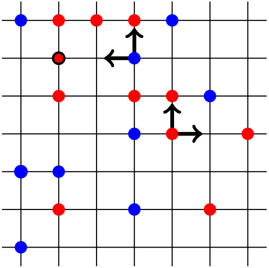

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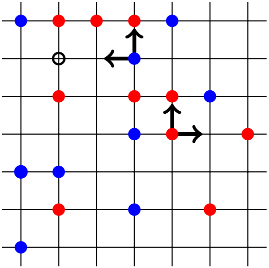

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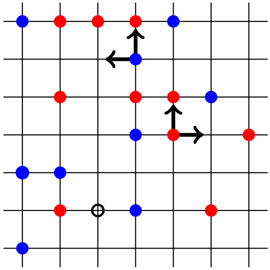

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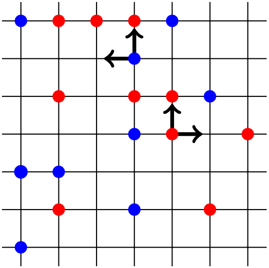

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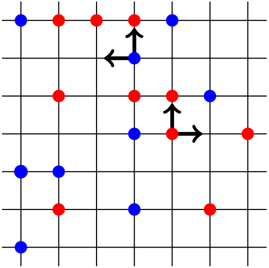

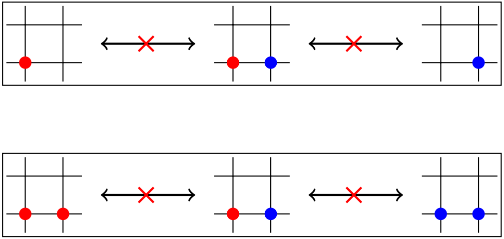

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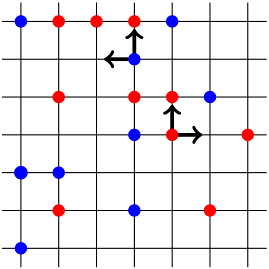

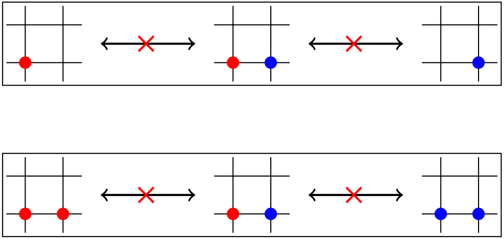

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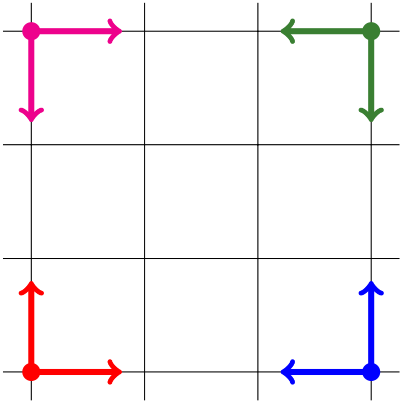
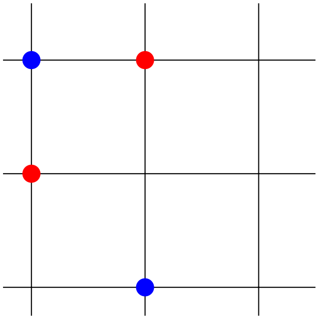
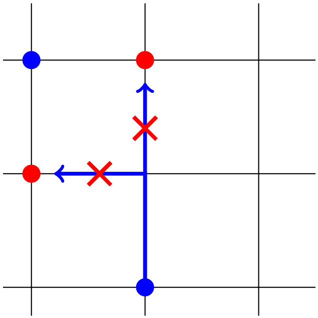
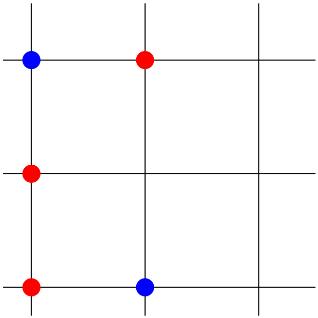
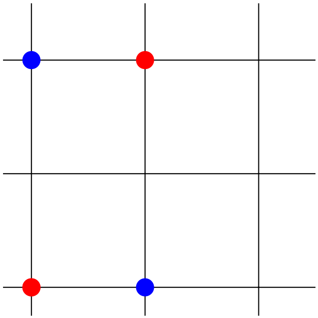
- Only $\circ \leftrightarrow {\color{red}\bullet}$ and $\circ \leftrightarrow {\color{blue}\bullet}$ transitions , no ${\color{blue}\bullet} \leftrightarrow {\color{red}\bullet}$ transitions!
- Reversible w.r.t. to product of μ_x giving $h \in \{{\color{red}\bullet}, {\color{blue}\bullet}\}$ with probability q_h and $\circ$ with probability p.

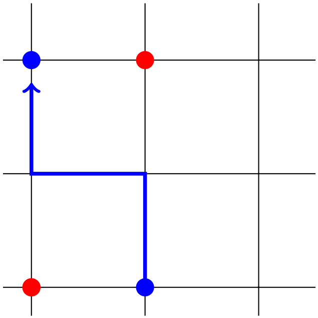












Ergodicity?

Spectral gap behaviour?

Theorem (Y.C. '22)

The multicolour East model on $\mathbb{Z}^2$ with

- *four colours is not ergodic.*
- *three or less colours has positive spectral gap.*

Spectral gap bounds

For simplicity: Two-colour East model with $q_{\color{blue}\bullet} < q_{\color{red}\bullet}$

Write $\theta_{\color{blue}\bullet} := \log_2(1/q_{\color{blue}\bullet})$, $\theta_{\color{red}\bullet} := \log_2(1/q_{\color{red}\bullet})$.

Theorem (Y.C. '22)

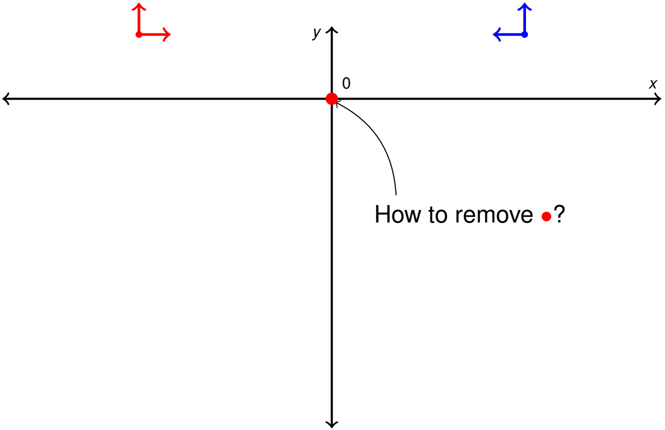
Fix $\Delta > 0$. If $p > \Delta$ we have

$$\lim_{q_{\color{blue}\bullet} \to 0} \frac{\gamma(\text{2-colour})}{\gamma_2(q_{\color{blue}\bullet})} = 1$$

$$\textit{If either} \quad \begin{cases} q_{\color{red}\bullet} = O(\theta_{\color{blue}\bullet}^{-3}), \textit{ i.e. “few } {\color{red}\bullet}\text{”,} \\ q_{\color{red}\bullet} \textit{ constant, i.e. “many } {\color{red}\bullet}\text{”.} \end{cases}$$

No frequent colour case

$q_{\color{blue}\bullet} \to 0$ and $q_{\color{red}\bullet} = O(\theta_{\color{blue}\bullet}^{-3})$



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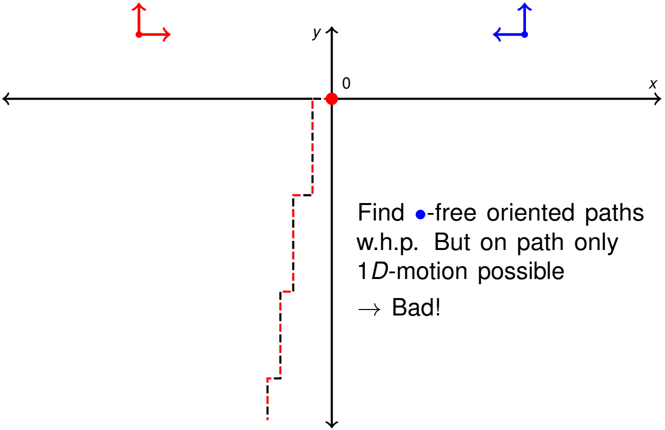
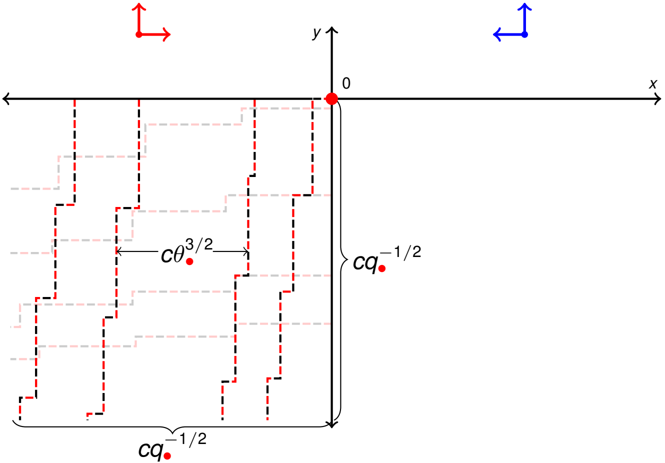


Find $\color{blue}\bullet$-free oriented paths w.h.p. But on path only 1D-motion possible

$\to$ Bad!

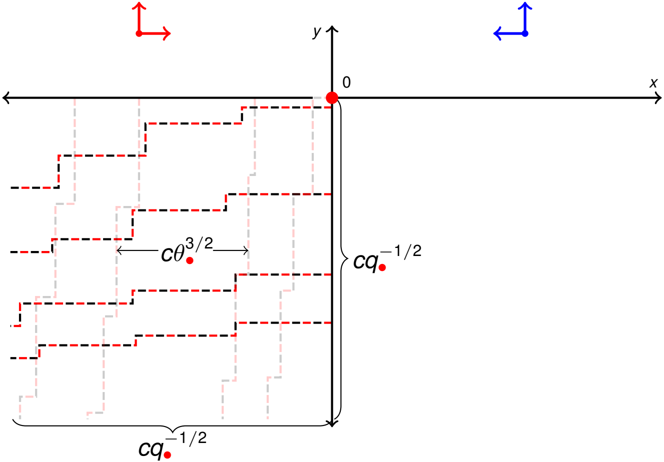
No frequent colour case

$q_{\color{blue}\bullet} \to 0$ and $q_{\color{red}\bullet} = O(\theta_{\color{blue}\bullet}^{-3})$



No frequent colour case

$q_{\bullet} \to 0$ and $q_{\bullet} = O(\theta_{\bullet}^{-3})$



No frequent colour case

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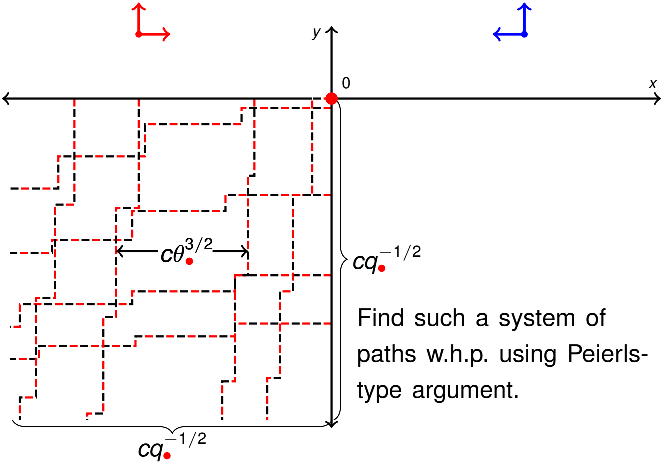
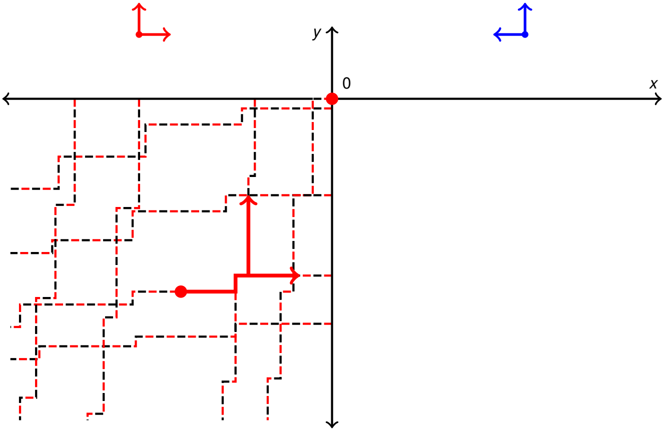


Find such a system of paths w.h.p. using Peierls-type argument.

No frequent colour case

$q_{\color{blue}\bullet} \to 0$ and $q_{\color{red}\bullet} = O(\theta_{\color{blue}\bullet}^{-3})$



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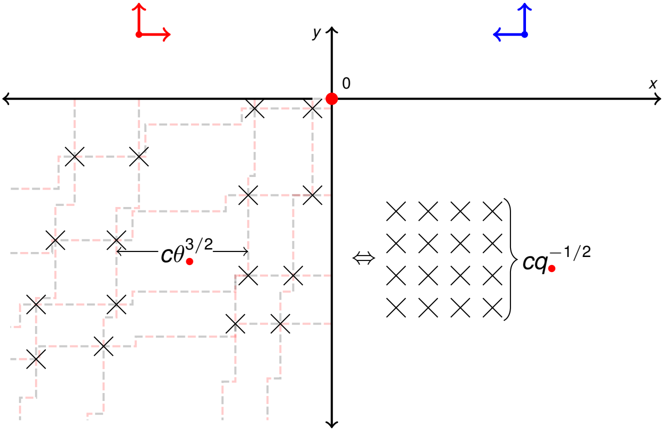
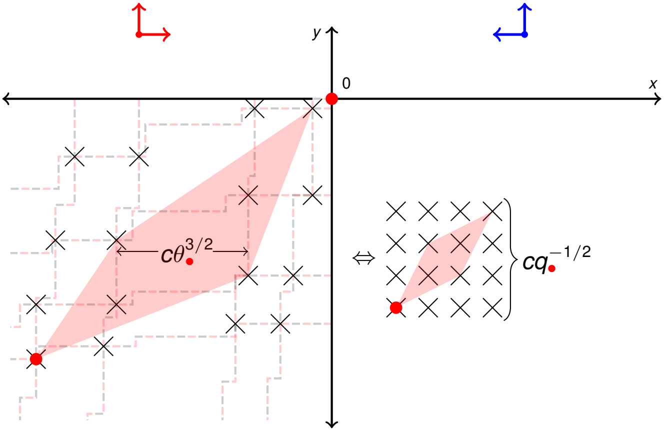


Competing dynamics: 2D on crosses, 1D between them.

No frequent colour case

$q_{\color{blue}\bullet} \to 0$ and $q_{\color{red}\bullet} = O(\theta_{\color{blue}\bullet}^{-3})$



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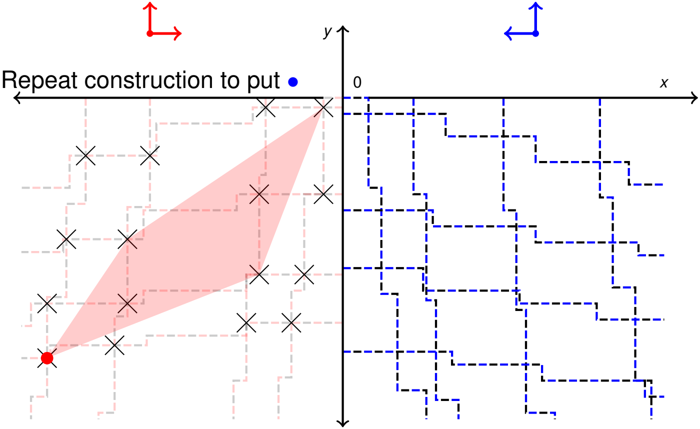

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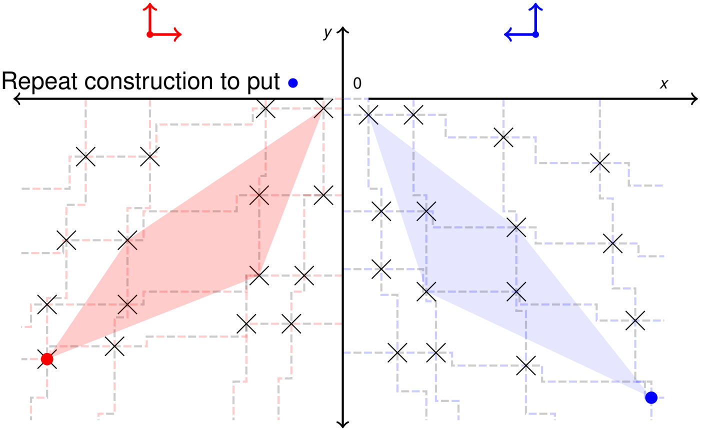
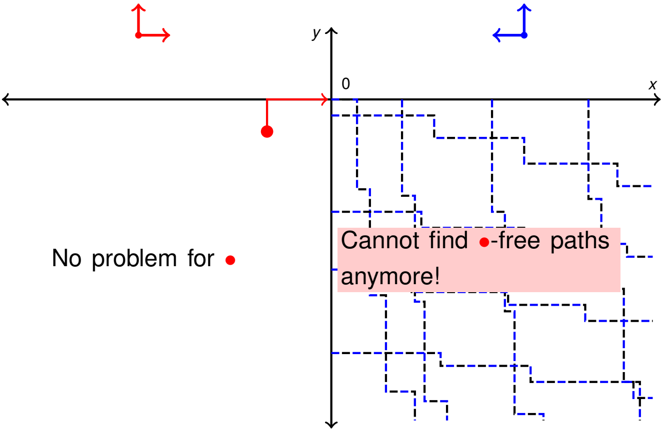


$$\implies \lim_{q_{\color{blue}\bullet}\to 0} \frac{\gamma(\text{2-colour})}{\gamma_2(q_{\color{blue}\bullet})} = 1$$

• frequent case

$q_{\bullet} \to 0$ and $q_{\bullet}$ constant



• frequent case

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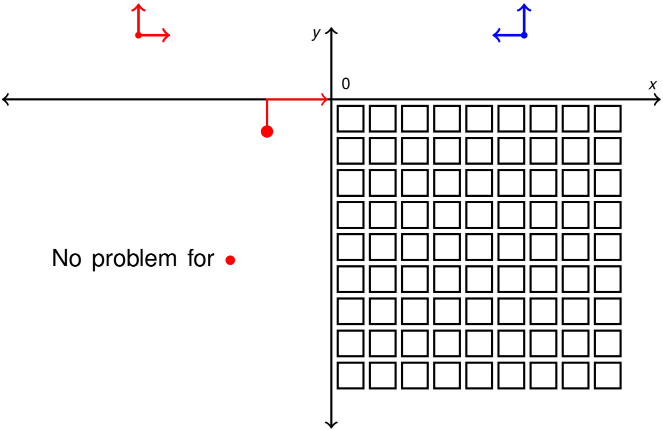
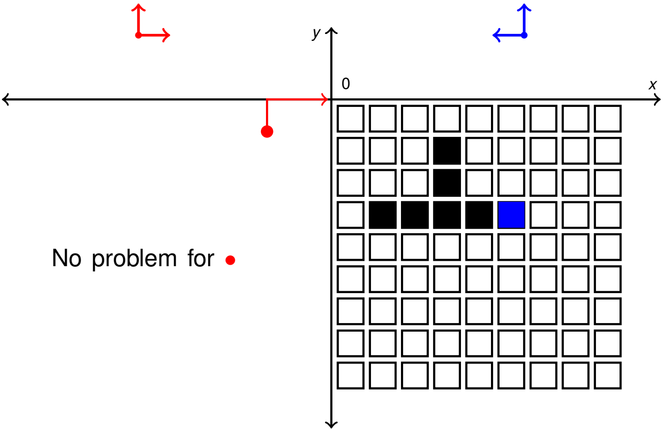


No problem for •

- Go to renormalized lattice.

• frequent case

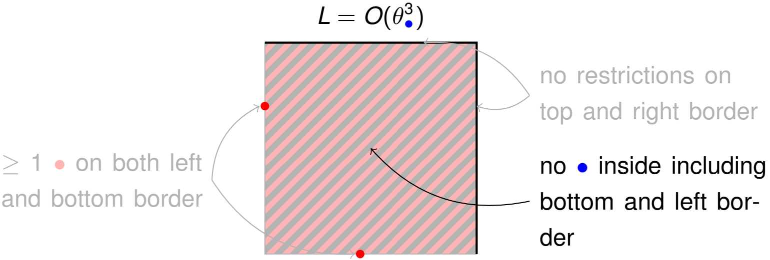
$q_{\color{blue}\bullet} \to 0$ and $q_{\color{red}\bullet}$ constant



- Go to renormalized lattice.
- Can we identify 'neutral' and 'blue' boxes on which we can repeat the previous construction?

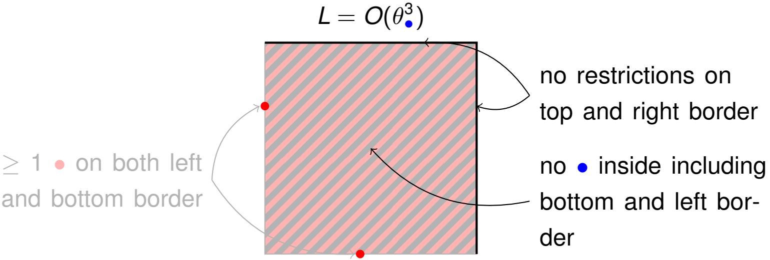
• frequent case

Construction of ‘neutral’ and ‘blue’ boxes



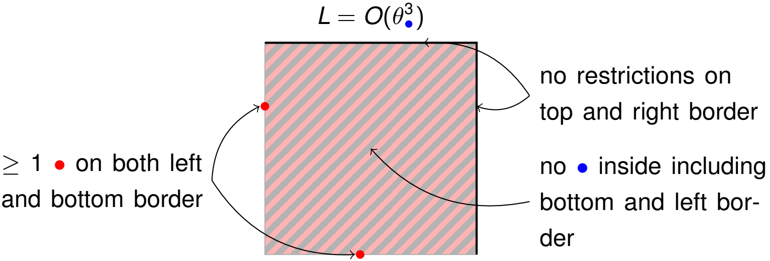
• frequent case

Construction of ‘neutral’ and ‘blue’ boxes



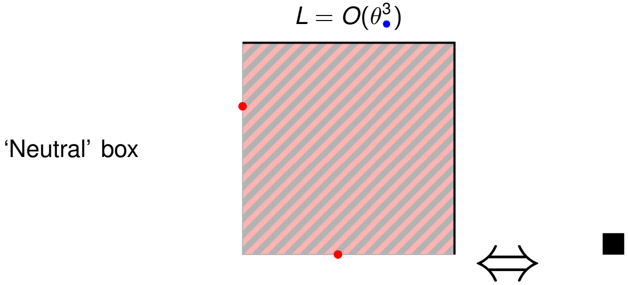
● frequent case

Construction of ‘neutral’ and ‘blue’ boxes



• frequent case

Construction of ‘neutral’ and ‘blue’ boxes



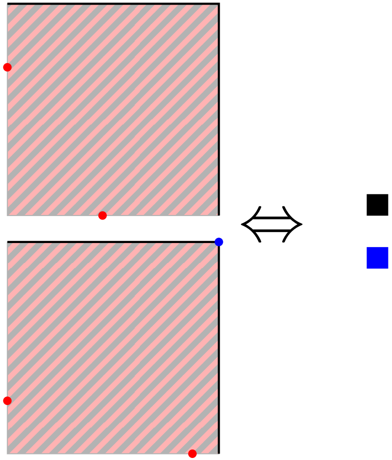
• frequent case

Construction of ‘neutral’ and ‘blue’ boxes

$L = O(\theta^3)$

‘Neutral’ box

‘Blue’ box

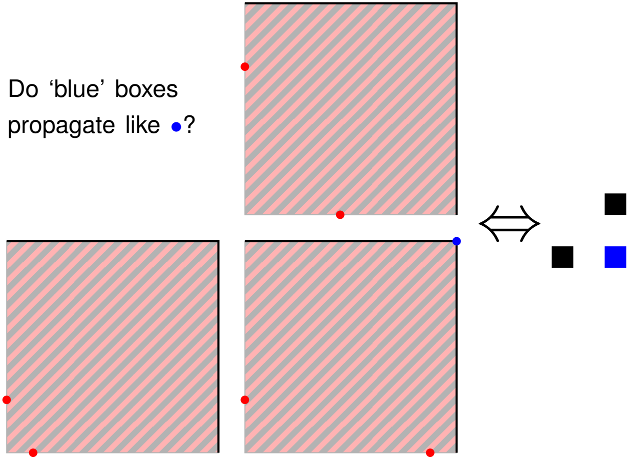


● frequent case

Construction of ‘neutral’ and ‘blue’ boxes

$L = O(\theta_{\bullet}^{3})$

Do ‘blue’ boxes propagate like ●?

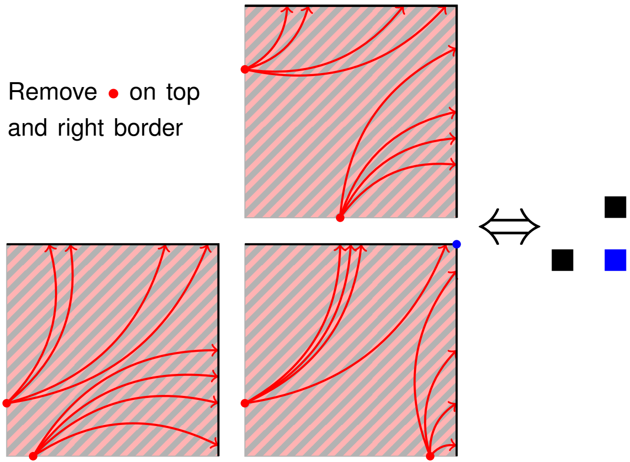


• frequent case

Construction of ‘neutral’ and ‘blue’ boxes

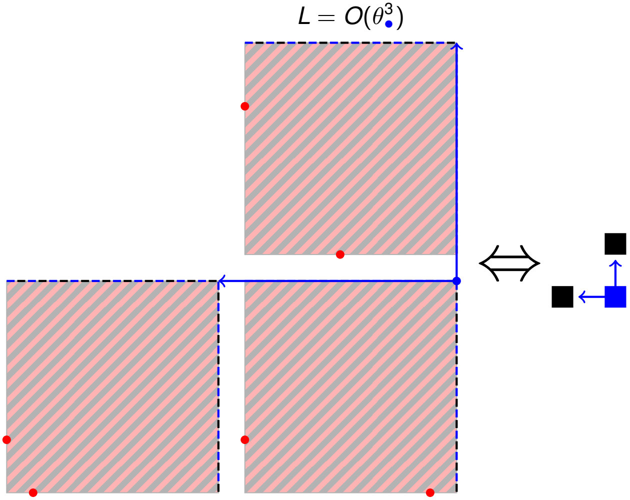
$L = O(\theta^3_{\bullet})$

Remove • on top and right border



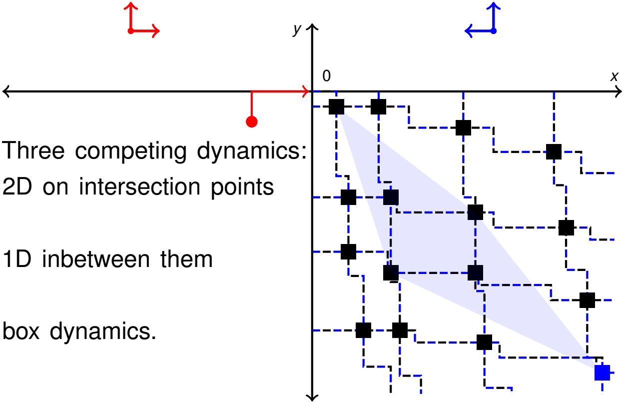
• frequent case

Construction of ‘neutral’ and ‘blue’ boxes



• frequent case

$q_{\bullet} \to 0$ and $q_{\bullet}$ constant



Three competing dynamics:

2D on intersection points

1D inbetween them

box dynamics.

$$\implies \lim_{q_{\bullet} \to 0} \frac{\gamma(\text{2-colour})}{\gamma_2(q_{\bullet})} = 1$$

Further results & open problems

- Positive spectral gap for $d \geq 3$ + given colour configurations.
  - Ergodicity landscape not fully explored for $d \geq 3$.
- Scaling of spectral gap in other two- and three-colour cases.
  - General spectral gap results unknown.

Thank you.

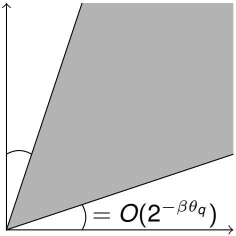
Between bulk and axes

Theorem (Y.C., F. Martinelli'22)

Fix $d \geq 2$.

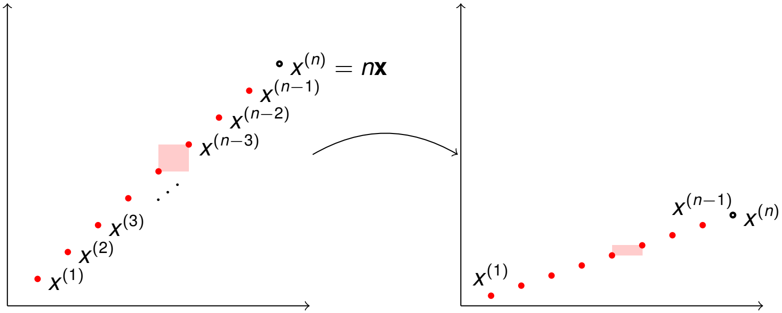
(B) *Let $0 < \beta < 1, \kappa \geq 1$ and let $\{\mathbf{x}(q)\}_{q \in (0,1)}$ be a family of unit vectors in $\mathbb{R}_+^d$ such that $\max_{i,j} \mathbf{x}_i(q)/\mathbf{x}_j(q) \leq \kappa 2^{\beta\theta_q}$. Then*

$$\limsup_{q \to 0} -\frac{2}{\theta_q^2} \log_2(v_{\min}(\mathbf{x}(q))) < 1.$$



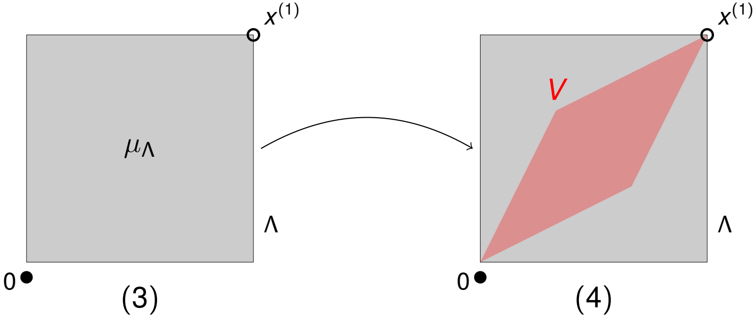
If $\mathbf{x}(q)$ approaches axes slowly enough we have $v_{\min}(\mathbf{x}) \gg v(\mathbf{e})$.

Proof of $\limsup_{q \to 0} -\frac{2}{\theta_q^2} \log_2(v_{\min}(\mathbf{x}(q))) < 1$



Proof of $\limsup_{q\to 0} -\frac{2}{\theta_q^2} \log_2(v_{\min}(\mathbf{x}(q))) < 1$

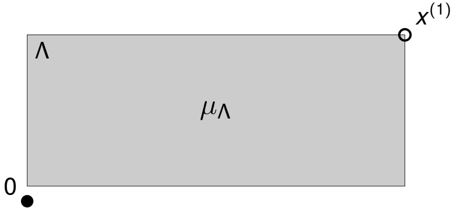
Before:



- Relate hitting time to spectral gap with min. b.c. on $V \subset \Lambda$
- RG-techniques: $\gamma(V) = 2^{-\theta_q^2(1\pm\varepsilon)/2d}$

Proof of $\limsup_{q\to 0} -\frac{2}{\theta_q^2}\log_2(v_{\min}(\mathbf{x}(q))) < 1$

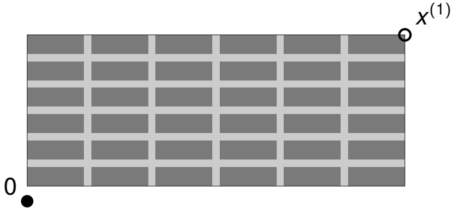
Now:



- Relate hitting time to spectral gap with min. b.c. on $V \subset \Lambda$
- $\gamma(\Lambda) = 2^{-\theta_q^2(1\pm\varepsilon)/2}$

Proof of $\limsup_{q\to 0} -\frac{2}{\theta_q^2}\log_2(v_{\min}(\mathbf{x}(q))) < 1$

Now:



- Relate hitting time to spectral gap with min. b.c. on $V \subset \Lambda$
- $\gamma(\Lambda) = 2^{-\theta_q^2(1\pm\varepsilon)/2}$

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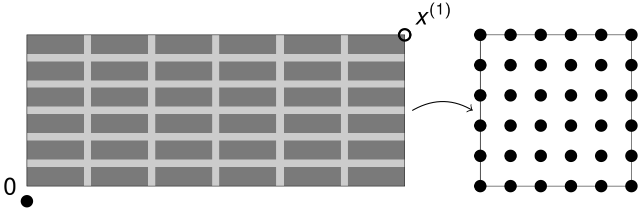

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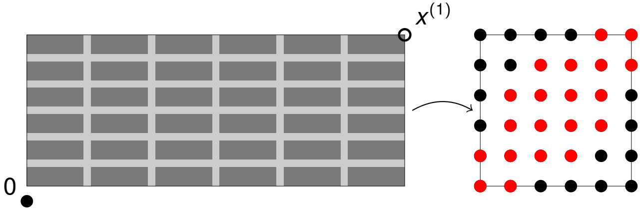

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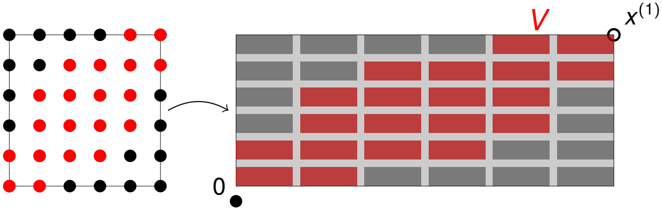


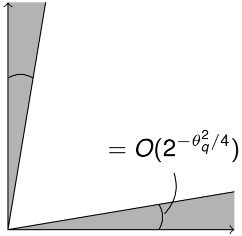
- Relate hitting time to spectral gap with min. b.c. on $V \subset \Lambda$
- $\gamma(V) = 2^{-\frac{\theta_*^2}{2d}(1\pm\varepsilon)}\gamma(\blacksquare) > 2^{-\theta_q^2(1\pm\varepsilon)/2}$.

Close to an axis

Theorem (Y.C., F. Martinelli'22)

(C) *Assume $d = 2$ and let $x(q)$ be such that $\max_{i,j} \mathbf{x}_i(q)/\mathbf{x}_j(q) \geq 2^{\theta_q^2/4}$. Then*

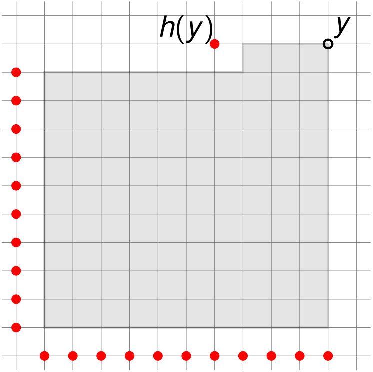
$$\lim_{q\to 0} -\frac{2}{\theta_q^2}\log_2(v_{\max}(\mathbf{x}(q))) = \lim_{q\to 0} -\frac{2}{\theta_q^2}\log_2(v_{\min}(\mathbf{x}(q))) = 1.$$



If $\mathbf{x} = \mathbf{x}(q)$ approaches one of the coordinate directions *fast enough*:

$$\begin{aligned} v_{\max}(\mathbf{x}) &= v_{\min}(\mathbf{x})^{1+o(1)} \\ &= v(\mathbf{e}_1)^{1+o(1)}. \end{aligned}$$

Proof of $v_{\max}(\mathbf{x}) = v(\mathbf{e}_1)^{1+o(1)}$



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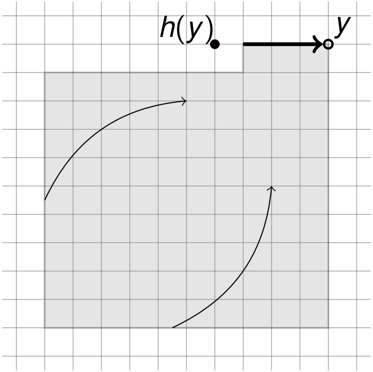
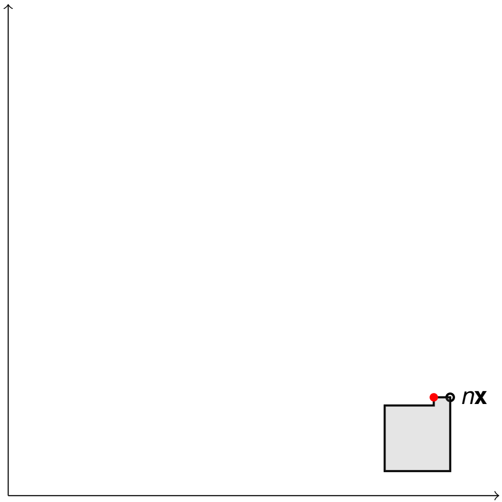


- $1d$-motion unaffected

Proof of $v_{\max}(\mathbf{x}) = v(\mathbf{e}_1)^{1+o(1)}$



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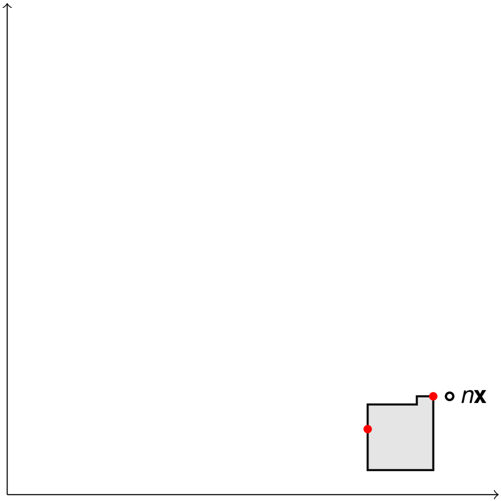

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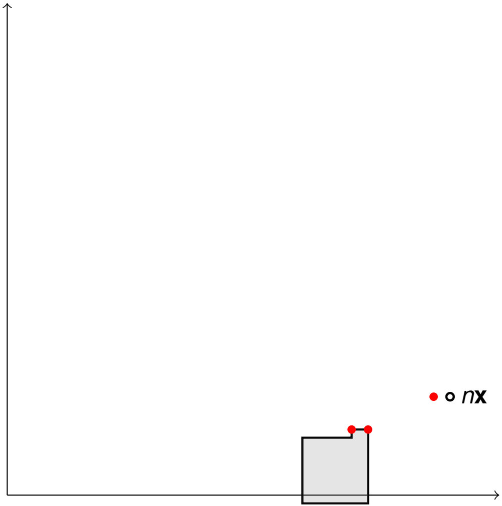

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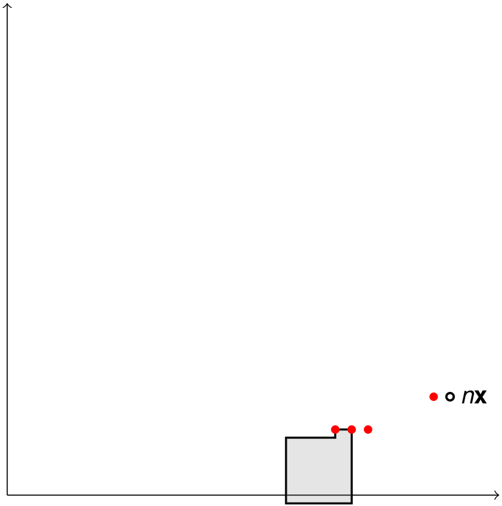

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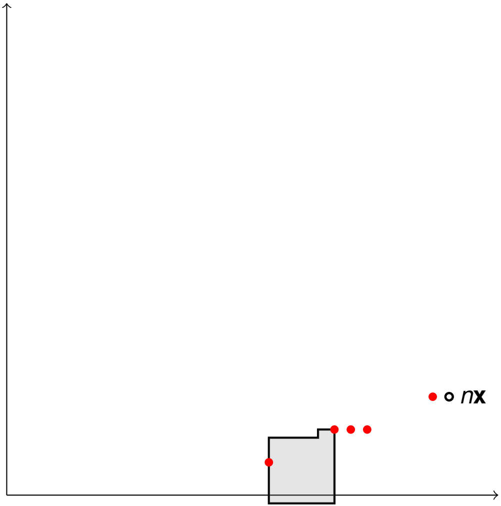

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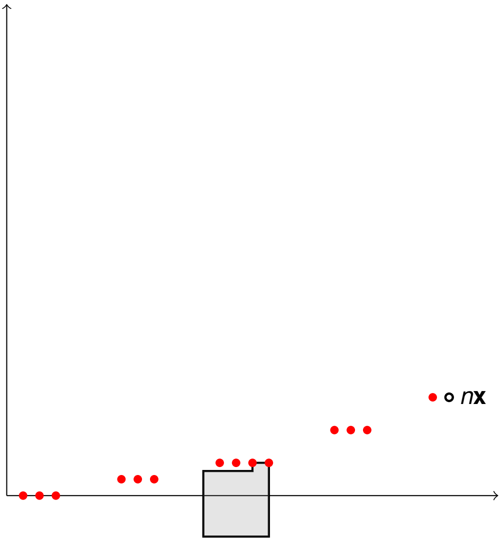

- Combinatiorally lower bound number of good points.

Cutoff

Let $\Lambda_n := \{0, \ldots, n\}^d$, $d_n(t) := \max_{\omega \in \Omega_{\Lambda_n}} \|\mathbb{P}^t_\omega - \mu_{\Lambda_n}\|_{TV}$ and consider

$$T^{(n)}_{\text{mix}}(\varepsilon) := \inf\{t > 0\colon d_n(t) \leq \varepsilon\}.$$

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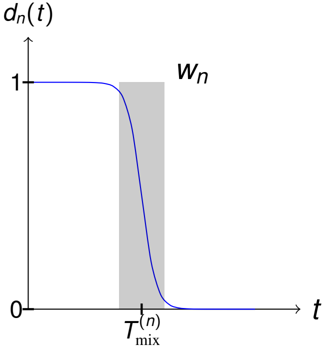


$$\lim_{\alpha \to -\infty} \liminf_{n \to \infty} d_n(T^{(n)}_{\text{mix}} + \alpha w_n) = 1$$

$$\lim_{\alpha \to \infty} \liminf_{n \to \infty} d_n(T^{(n)}_{\text{mix}} + \alpha w_n) = 0.$$

Mixing behind front

Theorem (Y.C., F. Martinelli'22)

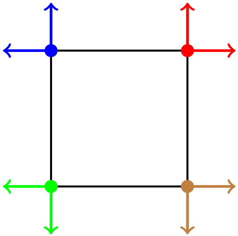
Fix $d \geq 2, 0 \leq \delta < 1$ and $\varepsilon > 0$. For $t > 0$ let $\nu_t^{\delta,\varepsilon}$ be the marginal on $\Omega_{\Lambda(\delta,\varepsilon,t)}$ of the law of the East process at time t with initial condition ω^*. Then,

$$\limsup_{\varepsilon\to 0}\limsup_{q\to 0}\limsup_{t\to\infty}\|\nu_t^{\delta,\varepsilon} - \mu_{\Lambda(\delta,\varepsilon,t)}\|_{TV} = 0 \quad \text{if } \delta > 0,$$

$$\liminf_{\varepsilon\to 0}\liminf_{q\to 0}\liminf_{t\to\infty}\|\nu_t^{\delta,\varepsilon} - \mu_{\Lambda(\delta,\varepsilon,t)}\|_{TV} = 1 \quad \text{if } \delta = 0.$$

Proof follows from front velocity bounds in first theorem and using CFM'15 to find that if every 'region' in a set has been infected, then equilibrium will spread 'quickly' in a region.

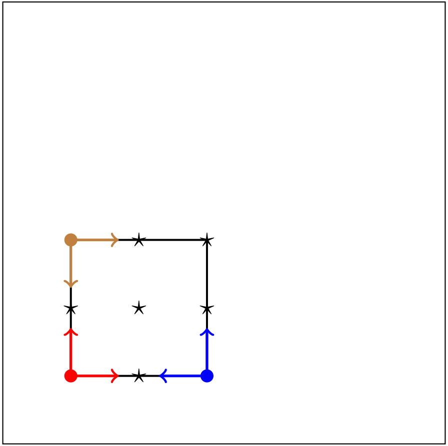
Non-ergodicity



- No legal transition possible out of this state.
- Appears almost surely if all vacancy-types have non-zero equilibrium density.

Ergodicity

- Ergodicity follows if almost surely there is a sequence of legal transitions starting from an equilibrium sampled state that puts any vacancy-type on $x \in \mathbb{Z}^2$.



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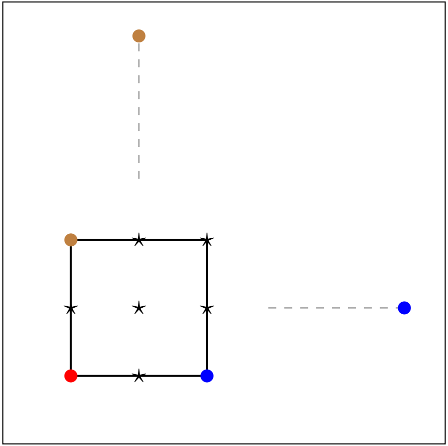

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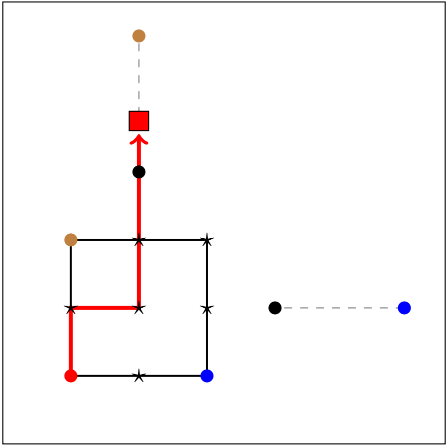

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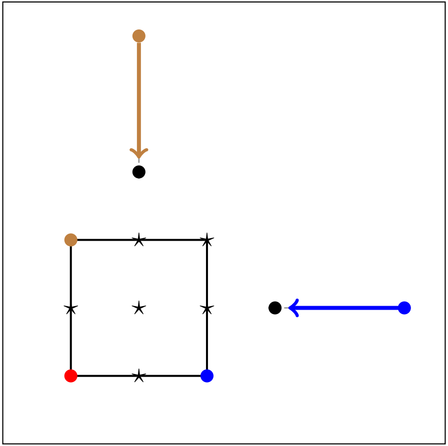

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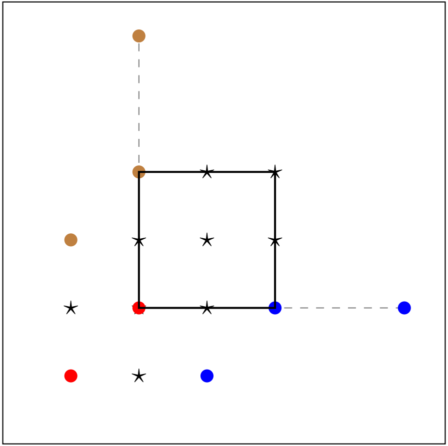

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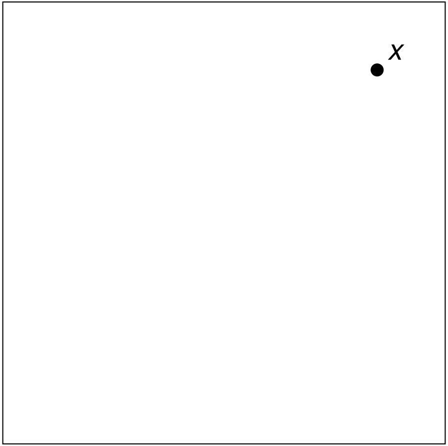

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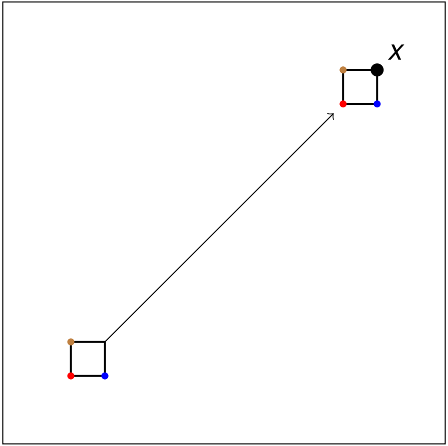

Theorem

Fix $\Delta > 0$ and consider a G-MCEM on $\mathbb{Z}^2$ with $|G| \in \{2,3\}$ and a valid parameter set $\mathbf{q}$ *such that $p > \Delta$. Then,*

$$\lim_{q_{\min}\to 0} \frac{\gamma(G;\mathbf{q})}{\gamma_2(q_{\min})} = 1 \qquad (1)$$

in the following cases.

- *Any 2-subset G and either one of the following conditions holds:*
  - (2.i) $\lim_{q_{\min}\to 0} q_{\max}\theta^3_{q_{\min}} = 0$,
  - (2.ii) $\lim_{q_{\min}\to 0} q_{\max}\theta^3_{q_{\min}} / \log_2(\theta_{q_{\min}}) = \infty$.
- *Any 3-subset $G \subset H_3$ and either one of the following conditions holds:*
  - (3.i) $\lim_{q_{\min}\to 0} q_{\max}\theta^3_{q_{\min}} = 0$,
  - (3.ii) $\lim_{q_{\min}\to 0} q_{\max}\theta^3_{q_{\mathrm{med}}} / \log_2(\theta_{q_{\min}}) = \infty$ *and* $\lim_{q_{\min}\to 0} q_{\mathrm{med}}\theta^6_{q_{\min}} = 0$,
  - (3.iii) *G is such that the vacancies associated to q_{med} and $q_{\max}$ share a propagation direction and* $\liminf_{q_{\min}\to 0} q_{\mathrm{med}} > 0$.