The multidimensional East model: a multicolour model and a front evolution problem

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Abstract

In this thesis we consider two problems related to the multidimensional East model on $\mathbb{Z}^d$, a well studied kinetically constrained model (KCM). KCM are interacting particle models in which the local configurations are updated with equilibrium only if the configuration in the neighbourhood of the update satisfies certain constraints. Usually KCM are defined on a local 0-1-state space where the 0s (or *vacancies*) are the facilitating states and the 1s (or *particles*) are the neutral ones. For the East model the constraints for the update at $x \in \mathbb{Z}^d$ require a vacancy on a smaller neighbour y in the lexicographic order.

The first problem is a front evolution problem as the equilibrium density q of the facilitating vertices vanishes. Starting with a unique unconstrained vertex at the origin, let $C(t)$ consist of those vertices which became unconstrained within time t and, for an arbitrary positive direction $\mathbf{x} \in \mathbb{R}^d_+$, let $v_{\max}(\mathbf{x}), v_{\min}(\mathbf{x})$ be the maximal/minimal velocities at which $C(t)$ grows in that direction. If $\mathbf{x}$ is independent of q, we prove that $v_{\max}(\mathbf{x}) = v_{\min}(\mathbf{x})^{(1+o(1))} = \gamma_d^{(1+o(1))}$ as $q \to 0$, where γ_d is the spectral gap of the process on $\mathbb{Z}^d$. We also analyse the case in which some of the coordinates of $\mathbf{x}$ vanish as $q \to 0$. In particular, for $d = 2$ we prove that if $\mathbf{x}$ approaches one of the two coordinate directions fast enough, then $v_{\max}(\mathbf{x}) = v_{\min}(\mathbf{x})^{(1+o(1))} = \gamma_1^{(1+o(1))} = \gamma_d^{d(1+o(1))}$, i.e. the growth of $C(t)$ close to the coordinate directions is dictated by the one-dimensional process. As a result the region $C(t)$ becomes extremely elongated inside $\mathbb{Z}^d_+$. Using these bounds on the front speed we identify an elongated subset $S(t) \subset C(t)$ that grows in t and which is mixing in $t \to \infty$. In fact, remarkably, these bounds on the front speed together with past results also imply a cutoff result for the East process on a box in $\mathbb{Z}^d$.

The second problem is a coarse-grained model of glass forming liquids introduced by Chandler and Garrahan [28] which is closely related to the East model. Instead of fixing a facilitation direction in the model, we consider multiple types of facilitating vertices that evolve on the same lattice, where each type behaves like a rotated version of the East process, e.g. in $d = 2$ one type requires a vacancy in the south-west neighbourhood of updating vertices, one south-east, one north-east and one north-west. The crux is that the neutral vertices, i.e. the particles in the East model, are shared for all types of facilitating vertex. We call this model the *multicolour East model* (MCEM). We prove that if the number of species is equal to the maximum amount of possible rotations the associated process, the MCEM process, is not ergodic. We then provide sufficient conditions so that the MCEM process is ergodic and the spectral gap positive in $\mathbb{Z}^d$. For example we show that in $d = 2$ any MCEM process with three types of facilitating vertices has a positive spectral gap. Further, for $d = 2$, we analyse the scaling of the spectral gap when the minimum density $q_{\min}$ of the facilitating vertex types tends to zero. We show sufficient conditions on the equilibrium distribution of the vertex types that the spectral gap tends to $\gamma_2(q_{\min})$ as $q_{\min} \to 0$. In particular, we show that this is also the case when vertices of the least frequent facilitating vertex type are surrounded by vertices of different types that inhibit their movement. We do this through a fine analysis, whereby the frequent vertex type move and remove each other in such a way as to clear the way for an effective two-dimensional motion of the infrequent types.

A novel technical ingredient is a detailed analysis of the asymptotics of a principal Dirichlet eigenvalue based on the renormalisation technique of Chleboun, Faggionato and Martinelli [13]. This analysis enters in both sets of results.

Print warning: When printing this thesis it is recommended to print at least the pages 24, 46, 71, 74, 75, 76, 93 and 100 in colour where we note that the page numbering starts at the mainmatter with Chapter 1 on the ninth page of the document so page 24 is the 32nd page of the document.

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Contents

Chapter 1

Introduction

In this chapter we seek to give a high-level introduction to and motivation for the problems treated in this thesis. A more formal introduction and contextualising of the main results can be found respectively in Chapter 3 for the front evolution problem and in Chapter 4 for the MCEM. This section may be safely skipped if no further motivation to read the thesis is needed.

1.1 Motivation and physical background

The liquid-glass transition is a fundamental problem of condensed matter physics that is still open. In the 1980s [24, 25, 48] physicists introduced an interacting particle system that evolves according to very simple rules that reflect some key features of glass forming liquids like non-Arrhenius behaviour of the relaxation time, dynamical heterogeneities and ergodicity breaking transitions among others. We call this interacting particle system a *kinetically constrained model* (KCM) and its evolution on $\mathbb{Z}^d$ can be described in two sentences:

With rate one the local configurations on the vertices of $\mathbb{Z}^d$ try to update with equilibrium. This is only possible if the configuration in the neighbourhood of the updating vertex satisfies certain constraints.

Usually KCM are defined on a local 0-1-state space where the 0s (*vacancies*) are the facilitating states and the 1s (*particles*) are the neutral ones. The simplicity of this models stems from the fact that it is reversible with respect to the product Bernoulli measure locally assigning 0 with probability q and 1 with $p = 1 - q$. Despite they exhibit the above mentioned unorthodox features making them interesting candidates for both the physical and mathematical community.

A classic example for a KCM is the *East process* on $\mathbb{Z}$ [31] where an update on x only happens if there is a vacancy on $x - 1$ and its $\mathbb{Z}^d$ counterpart, the multidimensional East model or East-like model, looking at any smaller vertex in the lexicographic order. The analysis of its properties and models derived from it are the subject of this thesis.

Other classic representatives are the FA-jf model on $\mathbb{Z}^d$ [24] where updates are legal if there are at least j vacancies in the neighbourhood of an updating vertex or the North-East model on $\mathbb{Z}^2$ [49] in which an update is legal if the south *and* west neighbours are vacancies.

Glass-forming liquids Let us make a small detour into the physics of glasses and explain how KCM fit into the picture, to motivate their mathematical analysis. In the liquid-solid transition, a liquid that is cooled below its melting temperature undergoes a phase transition into a new equilibrium state by arranging its molecules in a crystalline structure. In the case of a viscous glass-forming liquid, when cooled below its melting temperature instead of crystallising the liquid will increase its viscosity by multiple orders of magnitude as it is getting cooled until finally reaching a state where movement almost completely is arrested. This final state is called a glass and can either be seen as an amorphous solid or a frozen liquid that will not reach its equilibrium on any reasonable time scale[6, 27, 45].

The intuition behind seeing KCM as models for the liquid-glass transition is based on the following idea: with decreasing temperatures we can assume the average particle density in the liquid to increase or equivalently the vacancy density (i.e. space between particles) to decrease. At the beginning of the cooling process, in a low particle density environment, molecules in a liquid are not inhibited by other molecules in their movement. As the liquid gets cooled the neighbourhoods of the molecules get more and more crowded until the density is so high that only very few molecules still have space to move so that we slowly approach a dynamically arrested state without undergoing a transition in the dynamics between the molecules, i.e. the dynamics are still essentially that of a liquid. In this interpretation, it is the free space between the molecules, that we call *vacancies*, that governs the dynamics at low-temperatures and we may relate the inverse temperature β to the average vacancy density by $q \approx e^{-\beta}$.

KCM as models for glass-forming liquids While it certainly sounds like KCM are a reasonable way to model what might happen in glass-forming liquids, any physical model is to be judged on its capability of reproducing experimental observations. A core characteristic of low-temperature glass-formers is how non-Arrhenius their relaxation is. One can calculate average relaxation times through $T_{\text{rel}} \sim e^{\beta \Delta E}$ where ΔE is the activation energy. In *strong glasses*, like the common window in our houses for example, relaxation is close to Arrhenius, i.e. ΔE is roughly temperature independent. For *fragile glasses* that is not the case [19, 28]. This breadth in the behaviour as $q \to 0$ is reflected in KCM. Indeed, while the FA-1f for example features Arrhenius behaviour, the East model is strongly non-Arrhenius as we find that $\Delta E \sim \beta$ [13].

Another interesting property of KCM is that despite their simple dynamics they can exhibit ergodicity breaking transitions as $q \to 0$. For the East model and FA-jf this is not the case as they are ergodic for any $j \leq d$ and $q \in (0, 1)$ (see [11]), but the North-East model on $\mathbb{Z}^2$ has a critical $q_c \in (0, 1)$ so that the process is not ergodic anymore for $q \leq q_c$ (see [36]). For further treatment of the relevance of ergodicity breaking in glasses see [34] and KCM see [52].

Further that any transition needs to be facilitated. On large islands of particles the cost (in time) of bringing a vacancy to the middle of the island to facilitate can scale prohibitively with the size of the island so while a region with a distribution of vacancies and particles may perform legal updates relatively freely other regions can be frozen for long periods of times. These dynamical heterogeneities are also a property exhibited by glass formers [6, 14, 20, 35] so that altogether KCM can be seen as an interesting toy model to model them.

KCM as interesting mathematical models and related works

Now that we might be convinced that KCM have merit as a physical model we come to the mathematical challenges KCM pose, making them an interesting research subject in the field of mathematics and thus justifying the existence of this thesis.

Ergodicity breaking As mentioned above there is an ergodicity breaking transition in KCM in e.g. the north-east model on $\mathbb{Z}^2$ or a choice of constraints called the spiral model [28], meaning that entire regions completely freeze [36]. This opens up research possibilities to study, for relatively simple models, persistence times at the critical density [11] or hitting times in the non-ergodic region [50], which in the ergodic-region is related to the relaxation time (which is ∞ in the non-ergodic region).

Unusual behaviour of spectral gap and universality For KCM in the ergodic region (which for FA-jf or East) the spectral gap is non-zero. In particular, it has been proved in great generality that when they are ergodic they are exponentially ergodic with a positive spectral gap. However, often the spectral gap decreases sharply in q. This is in part due to the large cooperative movements necessary in the small q regime to facilitate single transitions. Recently a large body of work has been produced that identifies universality classes of models that exhibit the same scaling in q of their spectral gaps. These classes originated in the study of bootstrap percolation: Bootstrap percolation is a cellular automaton with the same constraints as KCM, but at every round of the cellular automaton unconstrained vertices switch to become vacancies and never become particles again (see [46] for a semi-recent review on just bootstrap percolation). Remarkably, the universality classes for infection times in bootstrap percolation can be refined and applied to the KCM case.

While we could cite some representative individual papers we refer the reader to [30] which is a recent PhD thesis on the topic that, aside from a seemingly exhaustive literature review on bootstrap percolation and KCM, also contains many of these new results and the open problems for this particular line of research.

Non-attractiveness KCM are not attractive models, i.e. vacancies do not necessarily favour more vacancies. Consider for example two East processes where one starts with more vacancies than the other, then this order is not maintained as the process evolves in time. This is because having more vacancies also facilitates legal updates that put particles. This makes out-of-equilibrium calculations much harder since attractiveness allows for a basic coupling between processes started from different states. Without such a coupling proving any kind of convergence to equilibrium becomes considerably more difficult. Nonetheless through the years some knowledge on the out-of-equilibrium case has been accumulated of which we present a non-exhaustive selection.

For $d = 1$, exponential convergence to equilibrium is shown for the classic KCM like East and FA-1f in [8, 10, 47]. In particular, front progressions and mixing results for the FA-1f and East model have been proven in [7, 9] and subsequently even cutoff was shown [21, 26].

For $d \geq 2$ much less is known and what is known seems to be limited to the East model or derivatives of it [14, 22, 23]. In the high temperature regime exponential convergence to equilibrium has been shown for the multidimensional East model [15, 40] and then even for the universality class containing the multidimensional East model in [41].

There is no front-progression or shape theorem analogous to the $d = 1$ results for $d \geq 2$. In [13] the speed of propagation along the axes and the diagonal of boxes with side length $1/q^{1/d}$ is found for the multidimensional East model. There is no prior work on finding a d-dimensional analogue for the sets behind the front that are mixing as in [7], a cutoff result as in [26] or a comprehensive shape theorem (ibid.).

Variations of KCM In KCM the constraints are always considered translation invariant throughout the lattice and the vacancies are independent. Interacting KCM, where a weak interaction between vacancies is added, have been considered in [12]. Recently KCM with random constraints have been studied in [50, 51]. In this, two models are chosen and at the beginning each vertex gets assigned the constraints from one of the two models. It is the first research of KCM in random environments and the open questions here are still abound. The considerations were for example only for $d \in \{2, 3\}$ and a select amount of models so that little is known for $d > 3$ and the other cases. Further, one could also consider a system with a dynamic random environment in which the constraints change in time.

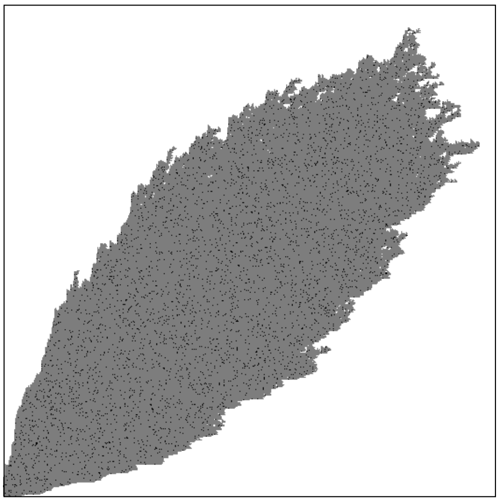

FIGURE 1.1 A simulation of the set $C(t)$ of vertices that had a legal update before some time t for $q = 0.04$ suggesting the existence of a limit shape.

1.2 Our contributions

Front evolution problem

Our first problem deals with a front progression of the multidimensional East process, identifies a mixing set behind the front and shows the mixing time cutoff phenomenon, so it lines up with previous research on the out of equilibrium East model and tackles the immediate open problems there.

Consider the East model on $\mathbb{Z}_+^d$ started from a vacancy-free state with only the origin unconstrained and consider the set of vertices $C(t) \subset \mathbb{Z}_+^d$ that had legal updates before time t. As mentioned above, an open problem is the question whether, in $d \geq 2$, we can find a shape $C_\infty \subset \mathbb{R}_+^d$ such that $C(t)/t \to C_\infty$ almost surely as $t \to \infty$ (see Figure 1.1). Let $v_{\max}(\mathbf{x})$, $v_{\min}(\mathbf{x})$ be the maximal/minimal velocities at which $C(t)$ grows in the direction of the unit vector $\mathbf{x}$.

In the vanishing q regime we show that if $\mathbf{x}$ is independent of the vacancy density q then $v_{\max}(\mathbf{x}) = v_{\min}(\mathbf{x}) = \gamma_d^{1+o(1)}$ with γ_d the spectral gap of the d-dimensional East process. If $\mathbf{x} = \mathbf{x}(q)$ approaches one of the coordinate axes as $q \to 0$ we distinguish two cases. If it approaches one of the axis slowly enough then we still find that $v_{\min}(\mathbf{x}) \gg \gamma_1^{1+o(1)} = \gamma_d^{d(1+o(1))}$, but if the approach is too quick, in $d = 2$, we prove that $v_{\max}(\mathbf{x}) = v_{\min}(\mathbf{x})^{(1+o(1))} = \gamma_1^{(1+o(1))}$. Thus, we don't identify the shape but identify directions where the propagation speed is maximized (i.e. γ_d) and directions in which it tends to the minimum (i.e. γ_1). Using this we identify a set $S(t) \subset C(t)$ that is extremely elongated around the main diagonal which mixes as $t \to \infty$, and remarkably note that the East process on an equilateral box exhibits a cutoff.

Multicolour East model

Recall that usually in KCM each vertex can only have two-states and the evolution is dictated by constraints that are fixed at the beginning of the process. An interesting problem is considering multicolour KCM in which there are multiple types of vacancies that come with their own specific constraints and only communicate with each other through the shared neutral state (which was the particle state before). An example of this, which is the other main subject of this thesis, is the *multicolour East model* (MCEM) in which there are multiple vacancy types that propagate individually like rotated multidimensional East model vacancies and the motion is mutually exclusive so that a vacancy can only be removed by a vacancy of the same type at the correct place.

As far as we know this is the first time that such a model is considered in the mathematical literature, but it is in fact inspired by a model introduced in the physical literature by Chandler and Garrahan in [27]. Their model has multiple kinds of vacancies that act on their own like rotated East model vacancies with the possibility of a ring that does not strictly respect the vacancy type limitations. It addresses the conceptual shortcomings of the East model that local movement inside the low-temperature glass-forming liquid globally has a single direction and that this direction cannot change.

We prove sufficient conditions for the positivity of the spectral gap, e.g. in $d = 2$ we prove that the spectral gap is positive for any MCEM with three vacancy types. Still in $d = 2$ we analyse the scaling of the spectral gap when the minimum density $q_{\min}$ of the vacancy types tends to zero. We prove sufficient conditions so that the spectral gap of the MCEM process with at most three vacancy types converges to $\gamma_2(q_{\min})$.

1.3 Structure of the thesis

This thesis is structured into two parts an expository part and a technical part. The expository part is split into four chapters. In Chapter 2 we formally construct the East process together with some past results. The next two expository chapters then introduce the two problems and our results.

In Chapter 3, we start with an introduction in Section 3.1 to the topic of shape theorems and the progress made towards finding front progression speeds. In Section 3.2 we present the propagation speed result in Theorem 1, the mixing set result in Theorem 3 and the cutoff result in Theorem 4. The proof of these results is contained in the second chapter of the technical part, Chapter 7. Understanding the proof only requires the reading of Chapter 6.

We start Chapter 4 in Section 4.1 by giving a more detailed physical motivation for introducing MCEM and give some heuristic arguments for ergodicity and spectral gap bounds for the two colour case on $\mathbb{Z}^2$. Section 4.2 contains the formal construction of the MCEM process. We end with the presentation of the results in Section 4.3, the first being an ergodicity result and the second an analysis of the spectral gap behaviour in given limiting cases for the equilibrium distribution. The proofs of the results are contained in the technical second part of the thesis. The ergodicity result is proved in Chapter 8 and can be read independently. The proof of the spectral gap behaviour result, Theorem 6, is contained in Chapter 9 and relies again on the results from Chapter 6.

Thus, apart from Chapter 6 that is shared for both problems the exposition and proof sections for the front evolution problem and for the MCEM can be read independently.

Chapter 5 serves as an overview for the employed techniques and a high-level guide to the main ideas behind the proofs contained in the technical part. It also naturally motivates Chapter 6 since Proposition 6.6, contained therein, solves an underlying question for both Theorem 1 and Theorem 6. It is recommended but not strictly necessary to read this chapter.

In Chapter 10 we close the thesis with a short review of the results presented in the thesis and discuss remaining conjectures and open questions resulting from them.

Part I

The two main problems

Chapter 2

The d-dimensional East process

The d-dimensional East process is the fundamental model behind both the front evolution problem and the multicolour East model so let us now come to the precise definition and some important past results.

2.1 Notation

- For $m, n \in \mathbb{N}$ with $m \le n$ we write $[m, n] := \{m, m+1, \dots, n\}$ and $[n] := [1, n]$ if $n \ge 1$.
- Let $\mathbb{R}^d_+ = \{x = (x_1, \dots, x_d) \in \mathbb{R}^d : \ x_i \ge 0 \forall i \in [d]\}$ and for any $x \in \mathbb{R}^d_+$ let $\lfloor x \rfloor \in \mathbb{Z}^d_+$ be such that $\lfloor x \rfloor_i = \lfloor x_i \rfloor$ for all $i \in [d]$. Unit vectors of $\mathbb{R}^d_+$ will be written in bold. Given $x, y \in \mathbb{Z}^d_+$ we will write $x \prec y$ iff $x_i \le y_i \ \forall i$, $x \prec V, V \subset \mathbb{Z}^d_+$, if $x \prec y \ \forall y \in V$, and $\|x - y\|_1 := \sum_i |x_i - y_i|$ for their ℓ_1-distance. We shall also write $x = 0$ to denote the origin of $\mathbb{Z}^d_+$. We use the letters x, y, z to denote vertices in $\mathbb{Z}^d$ and subscripts i, j, k to denote the components $(x_1, \dots, x_n)$.
- We use $\mathcal{B}$ to denote the canonical basis of $\mathbb{Z}^d$ comprised of the vectors $\mathbf{e}_1 = (1, 0, 0, \dots, 0)$, $\mathbf{e}_2 = (0, 1, 0, \dots, 0)$,..., $\mathbf{e}_d = (0, 0, \dots, 0, 1)$ and use $\cdot$ for the canonical dot product so that $x \cdot \mathbf{e}_i = x_i$.
- For $\Lambda \subset \mathbb{Z}^d$ we define its *oriented boundary* as $\partial_\downarrow \Lambda := \{x \in \mathbb{Z}^d \setminus \Lambda \colon x + \mathbf{e} \in \Lambda \text{ for some } \mathbf{e} \in \mathcal{B}\}$ and if $\Lambda \subset \mathbb{Z}^d_+$ we define its *positive oriented boundary* as $\partial^+_\downarrow \Lambda := \{x \in \mathbb{Z}^d_+ \setminus \Lambda \ : \ x + \mathbf{e} \in \Lambda \text{ for some } \mathbf{e} \in \mathcal{B}\}$.
- Given integers $(L_1, \dots, L_d)$ the set $\Lambda = \prod_{i=1}^d \{0, \dots, L_i\}$ will be called the *box* with side lengths $(L_1, \dots, L_d)$. We will write x_Λ for the vertex $(L_1, \dots, L_d)$. Notice that $\partial^+_\downarrow \Lambda = \emptyset$. Given a box Λ with side lengths $(L_1, \dots, L_d)$ the set $x + \Lambda$ will be called the *box* with side lengths $(L_1, \dots, L_d)$ and origin at x. Unless otherwise specified a box will always have its origin at $x = 0$.
- Ω_Λ will denote for the product space $\{0, 1\}^\Lambda$ endowed with the product topology of the discrete topology on $\{0, 1\}$. We will write $\omega_x \in \{0, 1\}$ for the state at $x \in \Lambda$ of the *configuration* $\omega \in \Omega_\Lambda$ and we will refer to the vertices of Λ where $\omega \in \Omega_\Lambda$ is equal to one (zero) as the *particles* (*vacancies*) of ω. If $V \subset \Lambda$ we will write $\omega \restriction_V$ for the restriction of $\omega \in \Omega_\Lambda$ to V. In particular we will write $\omega \restriction_V = 1$ if $\omega(x) = 1 \ \forall \ x \in V$. We use the Greek letters ω, σ, η to denote configurations in Ω_Λ.
- For any $\Lambda \subset \mathbb{Z}^d$, a configuration $\sigma \in \Omega_{\partial_\downarrow \Lambda}$ and for $\Lambda \subset \mathbb{Z}^d_+$ a configuration $\sigma \in \Omega_{\partial^+_\downarrow \Lambda}$ will be referred to as a *boundary condition for* Λ. If σ contains no particles it will be referred to as *maximal boundary condition*. If Λ is a box with side lengths in $\mathbb{Z}_+ \cup \{\infty\}$ and origin at x, we call σ a

minimal boundary condition if the only vacancy in σ is in $\{x - \mathbf{e} \colon \mathbf{e} \in \mathcal{B}\}$. Finally, for any given boundary condition $\sigma \in \Omega_{\partial_\downarrow \Lambda}$ and $\omega \in \Omega_\Lambda$, we will write $\sigma \cdot \omega \in \Omega_{\partial_\downarrow \Lambda \cup \Lambda}$ for the configuration equal to σ on $\partial_\downarrow \Lambda$ and to ω on Λ (respectively with $\partial_\downarrow^+ \Lambda$ for $\Lambda \subset \partial_\downarrow^+$).

- Given $\Lambda \subset \mathbb{Z}_+^d$ we will write μ_Λ for the product Bernoulli(p) measure on Ω_Λ and $\mu_\Lambda(f), \mathrm{Var}_\Lambda(f)$ for the average and variance of $f : \Omega_\Lambda \mapsto \mathbb{R}$ w.r.t. μ_Λ.
- Constants, when not immediately relevant to the results being proved may change from line to line without explicitly mentioning it, so for example we might write

$$\ell \cdot 2^{\theta_q^2 + \kappa \theta_q} = 2^{\theta_q^2 + \kappa \theta_q}$$

if ℓ is of the order 2^{θ_q} instead defining a new constant $\kappa' > \kappa$.

2.2 Construction and main properties

Given $\Lambda \subset \mathbb{Z}^d,\ \sigma \in \Omega_{\partial_\downarrow \Lambda}$ and $\omega \in \Omega_\Lambda$, define the *constraint* $c_x^{\Lambda,\sigma}(\omega)$ at $x \in \Lambda$ as

$$c_x^{\Lambda,\sigma}(\omega) = \begin{cases} 1 & \text{if } \exists \mathbf{e} \in \mathcal{B} : x - \mathbf{e} \in \partial_\downarrow \Lambda \cup \Lambda \text{ and } (\sigma \cdot \omega)_{x - \mathbf{e}} = 0, \\ 0 & \text{otherwise.} \end{cases}$$

If ω is such that $c_x^{\Lambda,\sigma}(\omega) = 1$ we say that x satisfies the constraints and if there is an x such that $\min_\omega c_x^{\Lambda,\sigma_x}(\omega) = 1$ we say that x is unconstrained.

Say that a function $f : \Omega_\Lambda \to \mathbb{R}$ is local if its value only depends on finitely many variables in Λ. We define the infinitesimal generator $\mathcal{L}_\Lambda^\sigma$ of the East process[1] on Λ with vacancy density parameter $q \in (0,1)$ and boundary configuration σ through its action on local functions f as

$$\begin{aligned} \mathcal{L}_\Lambda^\sigma f(\omega) &= \sum_{x \in \Lambda} c_x^{\Lambda,\sigma}(\omega) \left[\omega_x q + (1 - \omega_x) p\right] \cdot \left[f(\omega^x) - f(\omega)\right] \\ &= \sum_{x \in \Lambda} c_x^{\Lambda,\sigma}(\omega) \left[\mu_x(f) - f\right](\omega), \end{aligned}$$

where ω^x is the configuration in Ω_Λ obtained from ω by flipping its value at x. We are not going into the details of how to build continuous-time processes out of the action of generators on local functions (see for example [36, 39] for that) and assume some familiarity with the subject as we are only going to recall some notions and results as they are useful to us. In particular, we are not going to differentiate between functions in the domain of $\mathcal{L}$ and local functions for which the action is well defined and use these terms interchangeably. Let $f \in L^2(\mu_\Lambda)$, we define the state $\omega(t)$ of the process at a time $t \in \mathbb{R}_+$ started at $\eta := \omega(0)$ through the probability semi-group $e^{t\mathcal{L}_\Lambda^\sigma}$ as

$$\mathbb{E}_\eta^{\Lambda,\sigma}(f(\omega(t))) := e^{t\mathcal{L}_\Lambda^\sigma} f(\eta)$$

and we write $\mathbb{P}_\eta^{\Lambda,\sigma}$ for the corresponding probability measure, i.e. the probability over $\mathcal{A}(\omega(t))$ if $f(\omega(t)) := \mathbb{1}_{\mathcal{A}(\omega(t))}$. If instead of a fixed configuration our starting state is distributed according to some measure ν on Ω_Λ we write

$$\mathbb{E}_\nu^{\Lambda,\sigma}(f(\omega(t))) = \sum_{\eta \in \Omega_\Lambda} \nu(\eta) \mathbb{E}_\eta^{\Lambda,\sigma}(f(\omega(t))).$$

[1] We adopt the convention of speaking of processes for general Λ, and of chains for finite Λ.

As the local constraint $c_x^{\Lambda,\sigma}(\cdot)$ does not depend on the state of the process *at* x, the process is reversible w.r.t. μ_Λ. In fact, thanks to the orientation of the constraints a stronger property holds [15, Section 3]. We say that the process is ergodic with stationary measure μ if for any $f \in L^2(\mu)$ and any starting state η with at least a vacancy we have

$$\lim_{t\to\infty} \mathbb{E}_\eta^{\Lambda,\sigma}(f(\omega(t))) = \lim_{t\to\infty} e^{t\mathcal{L}_\Lambda^\sigma} f(\eta) = \mu(f)$$

in $L^2(\mu)$.

Remark 2.1. For $d \geq 2$ and any integer $d' \in [1, d-1]$ the projection of the East process on $\mathbb{Z}^d$ onto $\mathbb{Z}^{d'} = \{x \in \mathbb{Z}^d : x_j = 0 \, \forall j > d'\}$ coincides with the East process on $\mathbb{Z}^{d'}$. Similarly, for any finite $V \subset \mathbb{Z}^d$ and any box $\Lambda \supset V$ the projection of the East process on $\mathbb{Z}^d$ onto V coincides with the same projection of the East process on Λ.

Graphical construction Instead of using infinitesimal generators, there is an explicit construction of the East process, which helps in getting a clearer idea of the dynamics. Consider a finite subset $\Lambda \subset \mathbb{Z}^d$ and associate to each vertex $x \in \Lambda$ a marked Poisson process with rate one. The marks are given by i.i.d. $\mathrm{Bernoulli}(p)$ variables on the local $\{0,1\}$-state space so that with probability p they give the state 1 and with probability q they give the state 0. Assume that at time t there is a ring of the Poisson process associated to $x \in \Lambda$. If the constraints are satisfied at x for the configuration $\omega(t-)$ at time a time $t-$ infinitesimally smaller than t, i.e. $c_x^{\Lambda,\sigma}(\omega(t-)) = 1$, we say that we have a *legal ring*, and otherwise that we have an *illegal ring*. If the ring is illegal, nothing happens. If the ring is legal, we replace the state $\omega_x(t)$ with the outcome of the $\mathrm{Bernoulli}(p)$ variable (i.e. the mark associated to that ring). We will frequently use this construction, especially the notion of legal rings, in the sequel.

It is not obvious that this construction results in the same process as the one induced by the infinitesimal generator. Further, we assumed that it is possible to know the state of the neighbours of x at time $t-$ to evaluate the constraints $c_x^{\Lambda,\sigma}(\omega(t-))$, which is an assumption that requires justification in the case of infinite Λ. Indeed, it is possible to show that the amount of rings on which a state $\omega_x(t-)$ depends is finite, so that $\omega_x(t-)$ is well defined. We omit the details of this proof and the discussion of the relation between both constructions of the process and refer the reader to [36].

Dirichlet form and spectral gap We end the construction with some central objects for the results of this thesis. For $f : \Omega_\Lambda \mapsto \mathbb{R}$ define the *Dirichlet form* or the quadratic form of $-\mathcal{L}_\Lambda^\sigma$ as

$$\mathcal{D}_\Lambda^\sigma(f) := \mu_\Lambda(-f\mathcal{L}_\Lambda^\sigma f) = \sum_{x\in\Lambda} \mu_\Lambda(c_x^{\Lambda,\sigma}\mathrm{Var}_x(f)),$$

where

$$\mathrm{Var}_x(f)(\omega) = pq(\nabla_x f)^2(\omega) = pq(f(1\cdot\omega_{\Lambda\setminus\{x\}}) - f(0\cdot\omega_{\Lambda\setminus\{x\}}))^2.$$

We then define the *spectral gap* through its variational characterisation as

$$\gamma^\sigma(\Lambda; q) = \gamma^\sigma(\Lambda) := \inf_{\substack{f\in\mathrm{Dom}(\mathcal{L}_\Lambda^\sigma)\\ f\neq\mathrm{const}}} \frac{\mathcal{D}_\Lambda^\sigma(f)}{\mathrm{Var}_\Lambda(f)} .$$

If $\Lambda = \mathbb{Z}^d$ we write $\gamma(\mathbb{Z}^d; q) = \gamma_d(q)$. We call any inequality of the form

$$\mathrm{Var}_\Lambda(f) \leq C\mathcal{D}_\Lambda^\sigma(f) \quad \forall f,$$

a *Poincaré inequality* and note that the inverse spectral gap (i.e. the *relaxation time*) is the best constant in the above inequality. A well known result then connects the spectral gap to ergodicity.

Theorem 2.2 ([38, Section IV, Theorem 4.13]). *The following statements are equivalent*

(i) The East model is ergodic with stationary measure μ.

(ii) 0 is a simple eigenvalue of $\mathcal{L}^\sigma_\Lambda$.

Thus, ergodicity is implied by showing that the spectral gap is strictly larger than 0. In fact, a strictly positive spectral gap even implies mixing with exponentially decaying correlations (see for example [11, 37])

$$\mathrm{Var}(e^{t\mathcal{L}^\sigma_\Lambda} f) \le e^{-2t\gamma^\sigma(\Lambda)}\mathrm{Var}(f).$$

Finding tight bounds on the spectral gap is one of the central interests in the study of KCM and is our main concern for the multicolour East model section of this thesis. Before introducing the bounds on the spectral gap we use for the East model we introduce two further bits of notation, first $\theta_q := \log_2(1/q)$ and we write $g(q) = O(h(q))$, $g(q) = \Theta(h(q))$, $g = o(1)$ to mean that there are constants c, C and δ such that $|g(q)| \le C|h(q)|$ for any $q < \delta$, $c|h(q)| \le |g(q)| \le C|h(q)|$ for $q < \delta$ and $g(q) \to 0$ as $q \to 0$ respectively. For the multidimensional East model on boxes we have precise bounds on the spectral gap for minimal and maximal boundary conditions.

Theorem 2.3 ([13, Theorem 1 and 2]). *Let Λ be an equilateral box of side length $L-1$ with $L \in (2^{n-1}, 2^n]$ and $n = n(q)$ such that $\lim_{q\downarrow 0} n(q) = +\infty$. We then have*

$$\gamma^{\sigma^{\max}}(\Lambda) = \begin{cases} 2^{-(n\theta_q - d\binom{n}{2})(1+o(1))} & : n \le \theta_q/d, \\ 2^{-\theta_q^2(1+o(1))/2d} & : \textit{else}, \end{cases}$$

$$\gamma^{\sigma^{\min}}(\Lambda) = \begin{cases} 2^{-(n\theta_q - \binom{n}{2})(1+o(1))} & : n \le \theta_q/d, \\ 2^{-\theta_q^2(1+o(1))/2} & : \textit{else}, \end{cases}$$

where $\sigma^{\max}$ is a maximal boundary condition and $\sigma^{\min}$ a minimal one. If $\Lambda = \mathbb{Z}^d$ we have $\gamma_d = 2^{-\theta_q^2/2d(1+o(1))}$.

Remark 2.4. To simplify the notation we adopt the following convention for dropping the various super- and subscripts in the sequel. In chapters and sections concerning the multicolour East model specifically, we leave out the explicit mention of Λ if $\Lambda = \mathbb{Z}^d$ and in this case there is no oriented boundary so we also omit the boundary conditions. Minimal boundary conditions are implied if we omit the explicit mention of the boundary conditions on boxes in $\mathbb{Z}^d$.

In the rest of the thesis, which deals with the East model as defined above we only consider the model on $\mathbb{Z}^d_+$ or subsets thereof with minimal boundary conditions, i.e. where the origin is unconstrained. Notice that in this case the positive oriented boundary $\partial^+_\downarrow\Lambda$ may be empty or $\sigma = 1$ for $\sigma \in \partial^+_\downarrow\Lambda$ and the assumption would still be that the origin is unconstrained (otherwise the model would not be ergodic). Minimal boundary conditions are again implied if we omit the explicit mention of the boundary conditions on boxes in $\mathbb{Z}^d$ with origin $x > 0$.

Chapter 3

A front evolution problem for the multidimensional East model

We now come to a more in-depth introduction to the problem of shape theorems, front speeds and mixing sets which will give the necessary context to fully appreciate the results that we present in Section 3.2. The results in this chapter have been (pre)published by Fabio Martinelli and the author in [16].

3.1 Introducing the problem

Let $\omega^* \in \Omega$ be the configuration with no vacancy and write $\tau_x, x \in \mathbb{R}_+^d$, for the hitting time of the set $\{\omega :\ \omega_{\lfloor x \rfloor} = 0\}$. More generally, for any $A \subset \mathbb{Z}_+^d$ we will write τ_A for the hitting time of the set $\{\omega :\ \omega\restriction_A \neq 1\}$. Given a unit vector $\mathbf{x} \in \mathbb{R}_+^d$, it is known [15, Theorem 5.1] that for any fixed $q \in (0,1)$

$$\mathbb{E}_{\omega^*}(\tau_{n\mathbf{x}}) = \Theta(n), \quad \text{as } n \to +\infty. \tag{3.1}$$

Further recall that the mixing time $T_{\text{mix}}^{(n)}(\varepsilon)$ on $\Lambda_n := \{0, \dots, n-1\}^d$ is defined as the smallest t such that

$$d_n(t) = \max_{\omega \in \Omega_{\Lambda_n}} \|\mathbb{P}_\omega^t(\cdot) - \mu_{\Lambda_n}\|_{TV} < \varepsilon$$

where $\|\cdot\|_{TV}$ is the total variation distance, $\mathbb{P}_\omega^t(\cdot)$ denotes the law at time t of the East process on Λ_n with initial condition ω (see [37] for more details on the mixing time). In particular, we write $T_{\text{mix}}^{(n)} = T_{\text{mix}}^{(n)}(1/4)$. Then we know that $T_{\text{mix}}^{(n)} = \Theta(n)$ (ibid.). In these $\Theta(n)$ terms there are constants hiding that are of interest to us, as they tell us how fast equilibrium propagates, so it is natural to define

$$\frac{1}{v_{\max}(\mathbf{x})} = \liminf_{n\to\infty} \frac{\mathbb{E}_{\omega^*}\Big(\tau_{n\mathbf{x}}\Big)}{n}, \qquad \frac{1}{v_{\min}(\mathbf{x})} = \limsup_{n\to\infty} \frac{\mathbb{E}_{\omega^*}\Big(\tau_{n\mathbf{x}}\Big)}{n},$$

and denote them as the maximal, respectively the minimal, front velocity in the direction of x. Using Equation (3.1), $0 < v_{\min}(\mathbf{x}) \le v_{\max}(\mathbf{x}) < +\infty$ for all $\mathbf{x}$.

In analogy with the classic *shape theorem* for first passage percolation discussed below we conjecture that $v_{\max}(\mathbf{x}) = v_{\min}(\mathbf{x}) := v(\mathbf{x})$ and in that case $v(\mathbf{x})$ represents the *front velocity in the direction* $\mathbf{x}$. Similarly, for any $t > 0$ we define the random set (see Figure 1.1)

$$C(t) = \{x \in \mathbb{R}_+^d :\ \tau_x \le t\},$$

and conjecture that there exists a compact subset $\hat{C} \subset \mathbb{R}_+^d$ such that

$$\forall \varepsilon > 0 \quad \lim_{t\to\infty} \mathbb{P}_{\omega^*}\Big((1-\epsilon)t\hat{C} \subseteq C(t) \subseteq (1+\epsilon)t\hat{C}\Big) = 1.$$

Remark 3.1. For $d \geq 2$, Remark 2.1 together with the law of large numbers in $d = 1$ imply that $v_{\max}(\mathbf{e}) = v_{\min}(\mathbf{e})\ \forall \mathbf{e} \in \mathcal{B}$. For all other directions both conjectures are still widely open.

In absence of a proof for the above conjectures it is natural to ask whether there is any information we can get on $v_{\max}$, $v_{\min}$ depending on $\mathbf{x}$. In particular, we investigate the physically relevant regime when $q \to 0$, which we recall is the low-temperature regime. Recalling the simulation results shown in Figure 1.1 we expect that $v_{\min}(\mathbf{x}) \gg v_{\max}(\mathbf{x}')$ when $\mathbf{x}$ points in a somewhat diagonal direction and $\mathbf{x}'$ points in a direction almost parallel to an axis. Further, one expects there to be a shift from a 'two-dimensional speed' to a 'one-dimensional speed' with $\mathbf{x}$ approaching an axis. Indeed, our first result, Theorem 1, proves bounds identifying directions clearly in the *one-dimensional mode* and directions in which propagation is clearly quicker.

We show that when $\mathbf{x}$ is independent of q we have $v_{\max}(\mathbf{x}) = v_{\min}(\mathbf{x})^{1+o(1)} = \gamma_d^{(1+o(1))}$ as $q \to 0$. If $\mathbf{x} = \mathbf{x}(q)$ slowly approaches an axis we have $\mathbf{v}_{\min}(\mathbf{x}) \gg \mathbf{v}(\mathbf{e}_1)$ as $q \to 0$ and if $\mathbf{x}$ approaches the axis quickly we get $v_{\max} = v_{\min}(\mathbf{x})^{1+o(1)} = v(\mathbf{e}_1)^{1+o(1)} = \gamma_1^{1+o(1)}$.

Above we said that $T_{\text{mix}}^{(n)} = \Theta(n)$. Given that the diagonal modes propagate with speed $\gamma_d \gg \gamma_1$ it begs the question whether we can find a subset $\Lambda_t \subset \mathbb{Z}_+^d$ that grows in time with speed γ_d and such that for ν_t the marginal of the East process on Λ_t at time t starting from the empty state we have

$$\lim_{t\to\infty} \|\nu_t - \mu_{\Lambda_t}\|_{TV} = 0. \tag{3.2}$$

Such a set Λ_t obviously could not include the axis as, let alone mixing[1], the propagation speed would not suffice. Finding such a set for $q \to 0$ is the object of Theorem 3.

Finally, let us recall the mixing time cutoff phenomenon:

Definition 3.2. We say that a Markov process on Λ_n exhibits *cutoff with a window of size $O(w_n)$* if $w_n = o(T_{\text{mix}}^{(n)})$ and

$$\lim_{\alpha\to-\infty} \liminf_{n\to\infty} d_n(T_{\text{mix}}^{(n)} + \alpha w_n) = 1,$$
$$\lim_{\alpha\to\infty} \liminf_{n\to\infty} d_n(T_{\text{mix}}^{(n)} + \alpha w_n) = 0.$$

In words the transition from an unmixed state, i.e. $\|\mathbb{P}_\omega^t(\cdot) - \mu_{\Lambda_n}\| \approx 1$, to a completely mixed state, i.e. $\|\mathbb{P}_\omega^t(\cdot) - \mu_{\Lambda_n}\| \approx 0$, happens at $T_{\text{mix}}^{(n)}$ in a time window of w_n. We refer the reader to [1, 18, 37] for more details on the cutoff phenomenon.

As we discuss further below, in $d = 1$ we know that the East process mixes in time linear in n with a cutoff of window $\sqrt{n}$. For $d > 1$ we only know that $T_{\text{mix}}^{(n)}$ is linear in n but not whether there is cutoff. This is the subject of Theorem 4 which proves a cutoff result for $d = 2$ on equilateral boxes with minimal boundary conditions for the same mixing time with a window smaller than $n^{2/3}$.

[1] We use the term *mixing* w.r.t. a set Λ to mean that the marginal ν_t of the East process at time t converges to the equilibrium marginal on Λ in the total variation distance, i.e. in this case Λ_t is mixing if Equation (3.2) holds.

3.1.1 Motivation

Shape theorem for first passage percolation

We start our exposition on the subject with the classic result for first passage percolation first given in [17] and present here a simplified version from [42] adapted to the first passage percolation case (see [5] for a recent review of first passage percolation). In this section let $\{T(z)\}_{z\in\mathbb{Z}_+^d}$ denote an i.i.d. family of positive weights distributed as an exponential random variable with parameter 1. Let $\Gamma(z_1,z_2)$ be the set of directed paths (going up and right) from z_1 to z_2 and $\Gamma(z_1)=\Gamma(0,z_1)$. Define

$$S(z,z')=\min_{\pi\in\Gamma(z,z')}\sum_{z''\in\pi}T(z''),\quad S(z)=S(0,z),\quad h(x)=\inf_{n\in\mathbb{N}}\frac{1}{n}\mathbb{E}(S(\lfloor nx\rfloor)).$$

If we consider $T(z)$ as the time it takes to go through the vertex z, then the model takes its name from the fact that $S(z)$ is the first time we could pass through z starting at the origin. The analogous set $C(t)$ for this model is then

$$C^{(fpp)}(t)=\{x\in\mathbb{R}_+^d\colon S(\lfloor x\rfloor)\le t\},\qquad C^{(fpp)}=\{x\colon h(x)\le 1\}.$$

The following shape theorem holds.

Theorem 3.3 (Proposition 2.1 [42])**.** *For any $\varepsilon>0$ there is a t_ε such that*

$$(1-\varepsilon)C^{(fpp)}\subset\frac{C^{(fpp)}(t)}{t}\subset(1+\varepsilon)C^{(fpp)},$$

almost surely for all $t>t_\varepsilon$.

Write $x\prec y$ if $x_i\le y_i$ for $i\in[d]$. The crucial property for the proof of this theorem is the *subadditivity* of $S(z)$, i.e. for any $z_1,z_2,z_3\in\mathbb{Z}_+^d$ $z_1\prec z_2\prec z_3$ we have

$$S(z_1,z_2)+S(z_2,z_3)\ge S(z_1,z_3).$$

Subadditivity combined with Kingman's subadditive ergodic theorem (see [33]) imply that

$$\lim_{n\to\infty}S(nz)/n\to h(z),$$

almost surely. The proof then requires a few more analytical steps that we omit here.

Subadditive quantities for the multidimensional East model When trying to tackle a shape theorem for the East model the natural approach would be to find a subadditive quantity representing the front. While $\mathbb{E}_{\omega^*}(\tau_{n\mathbf{x}})$ might not be subadditive, $\hat{T}(z_1,z_2):=\sup_{\omega,\omega_{z_1}=0}\mathbb{E}_\omega(\tau_{z_2})$ is for $z_1\prec z_2$. Indeed, by the strong Markov property, for $z_1<z_2<z_3$

$$\begin{aligned}\hat{T}(z_1,z_3)&=\sup_{\omega,\omega_{z_1}=0}\mathbb{E}_\omega(\tau_{z_3}(\mathbb{1}_{\tau_{z_2}<\tau_{z_3}})+\tau_{z_3}\mathbb{1}_{\tau_{z_2}\ge\tau_{z_3}})\\&\le\sup_{\omega,\omega_{z_1}=0}\mathbb{E}_\omega(\tau_{z_3}\mathbb{1}_{\tau_{z_2}<\tau_{z_3}})+\sup_{\omega,\omega_{z_1}=0}\mathbb{E}_\omega(\tau_{z_2})\\&\le\hat{T}(z_2,z_3)+\hat{T}(z_1,z_2),\end{aligned}$$

where we used the strong Markov property and that $\omega(\tau_{z_2})$ has a 0 in z_2. Thus, we get that $\hat{v}(\mathbf{x})^{-1}:=\lim_{n\to\infty}\sup_{\omega,\omega_0=0}\mathbb{E}_\omega(\tau_{n\mathbf{x}})/n$ exists. Clearly $v_{\max}(\mathbf{x})\le\hat{v}(\mathbf{x})$, but this is also the extent of information on the front we get through this approach.

Subadditivity through attractiveness Trying to find a subadditive relation without the supremum in the setting of more general interacting particle systems (see [4, 32] for example) often leads to arguments based on attractiveness (see [32] for a recent example of using subadditivity without attractiveness). Consider for example the frog model from [4] defined as follows: In the starting state we have a sleeping particle on each vertex in $\mathbb{Z}^d$ apart from some $x \in \mathbb{Z}^d$ which is active. The active particle performs a simple random walk and when it lands on a vertex with a sleeping particle, it wakes the sleeping particle on the vertex up so the newly active particle also starts to perform an independent simple random walk. Let $T(x, y)$ be the time it takes to wake the particle at y up starting with a single active particle at x, then we have

$$T(x, z) \leq T(x, y) + T(y, z).$$

Indeed, if z is reached before y this is obvious and if y is reached first consider that the process started with a single active particle at x, at the time it reaches y has many more active particles than the process started with a single active particle in y. By using the same jumps for both processes, what we call the *basic coupling*, it is easy to see that the above inequality holds.

The property that $T(y, z)$ being larger than $T(x, z) - T(x, y)$ is what we call *attractiveness*[2]. For the East model we do not have attractiveness, as having more vacancies, implies that legal updates that put particles get more likely, which in turn implies that more vacancies at the beginning might be detrimental to the progression of the front. In fact, all KCM are not attractive rendering this approach unusable.

Shape theorem and cutoff for the one-dimensional East model

In $d = 1$ we have a shape theorem for the East model for which the concept of the *distinguished zero*, first introduced in [2], is a central notion. Consider the East process on $\mathbb{Z}$ started from some state $\omega(0)$ with a vacancy on some $\zeta_0 \in \mathbb{Z}$ that we call the distinguished zero. Let τ_x be the first time, the vacancy on ζ_0 gets removed, then this implies that there is a vacancy on $\zeta_0 - \mathbf{e}$ that made the ring legal. We then move the distinguished zero ζ_t at time t to $\zeta_0 - \mathbf{e}$. Thus, ζ_t is a west-moving function in time such that $\omega_{\zeta_t} = 0$ for all t. In particular, the following important property holds:

Lemma 3.4 ([2, Lemma 4] or [12, Lemma 3.5])**.** *Fix an interval $V_0 = (x_0, x_1]$. Suppose that for the starting state $\omega(0)$ we have $\omega_{x_0}(0) = 0$ and that $\omega_{V_0}(0) \sim \mu_{V_0}$. If $\zeta_0 = x_0$ then the distribution of $V_t = (\zeta_t, x_1]$ is μ_{V_t} for any time $t \geq 0$.*

Sketch of proof. The proof is inductive: By stationarity the distribution on V_t is always μ_t since the boundary condition is fixed at 0. Whenever the distinguished zero moves, the newly added vertex to V_t is distributed like a Bernoulli(q) random variable independent of the config on V_t since there was just a legal update on it and hence the distribution on V_t is still that of a product Bernoulli measure. $\square$

Using the distinguished zero, Blondel was able to show in [7] that far behind the front the process is mixing. Additionally to the distinguished zero, Blondel also introduced the process as seen from the front, defined as the regular East process shifted so that the right-most vacancy ξ_t is always at the origin. By constructing a non-trivial coupling for ξ_t the existence of an invariant measure ν and ergodicity of the process as seen from the front is shown implying the existence of a shape, i.e. that $v_{\max} = v_{\min} = v$. In fact, a law of large numbers for ξ_t is shown.

This is the first proof of a shape theorem for KCM, but little was known of the invariant measure of the process seen from the front and as a consequence also of v. As mentioned above, in [26], Ganguly,

[2] More precisely, it is a consequence of what one usually calls attractiveness, the formal definition of attractiveness itself we leave out here, see for example [38] for that.

Lubetzky and Martinelli then proved a precise CLT for ξ_t showing that the East process exhibits cutoff in a window of $\sqrt{n}$. We explicitly recall the theorem here as it is often cited later.

Theorem 3.5 ([26, Theorem 2]). *There is a v such that the East process on Λ_n with parameter $0 < q < 1$ exhibits cutoff at $v^{-1}n$ with window $\sqrt{n}$, i.e. for $0 < \varepsilon < 1$ and n large enough we have*

$$T^{(n)}_{\text{mix}}(\varepsilon) = v^{-1}L + O\Big(\Phi^{-1}(1-\varepsilon)\sqrt{n}\Big)$$

where Φ is the c.d.f. of $\mathcal{N}(0,1)$ and the implicit constant in the $O(\cdot)$ depends only on q.

Generalising the to d-dimensions The concept of the distinguished zero is a concept that may be generalised to d-dimensions. This was done in [15] and used to prove a sort of convergence to equilibrium result. Notice that the proof of Lemma 3.4 critically relies on the distribution right of ζ_t to be independent of the distribution left of it. In $d \geq 2$ this is obviously not the case anymore so that to recover an analogous result to Lemma 3.4 (see [15, Proposition 3.5]) the authors had to condition on all the information that could influence the path of the distinguished zero to maintain equilibrium. The main contribution the distinguished zero gives in $d = 1$ is that there is an interval where we reasonably can expect to find vacancies. The distinguished zero does not give an analogue for this in $d \geq 2$. Instead we use that if we start with a vacancy at origin, for any time t there is an $x \in [-\ell, 0]^d$, that spends at least ℓ^{-d}/t time in the vacancy state (see [15, Remark 4.4 and Corollary 4.2] or Equation (7.2) below).

Further, also the process as seen from the front promises to be a much more delicate object to define in $d > 1$. In $d = 1$ it suffices to shift the East process but in $d > 1$ there is no obvious shift anymore as the front is now a $d - 1$ dimensional hyperplane rather than a point.

3.1.2 Previous result on scale $O(2^{\theta_q/d})$

In [13], Chleboun, Faggionato and Martinelli introduced an idiosyncratic way of partitioning $\mathbb{Z}^d$, the Knight lattice, together with a bottleneck construction which allowed them to prove Theorem 2.3 using a renormalisation group technique. The Knight lattice is heavily used in Proposition 6.6 in Chapter 6 and the bottleneck in Chapter 7 so we postpone their introduction here. They further prove the following bound, more closely resembling what we are looking for.

Theorem 3.6 ([13, Theorem 3]). *Consider the equilateral box Λ with side length $L \in (2^{n-1}, 2^n]$ and $n = n(q)$ with $\lim_{q\to 0} n(q) = \infty$. Then, as $q \to 0$*

$$\mathbb{E}_{\omega^*}(\tau_{x_\Lambda}) = 2^{n\theta_q - d\binom{n}{2} + O(\theta_q \log(\theta_q))},$$

for all $n \leq \theta_q/d$.

The proof relies on the above mentioned bottleneck and capacity methods combined with a sophisticated combinatorial analysis. This in particular does not give bounds on $v_{\max}(\mathbf{1})$ or $v_{\min}(\mathbf{1})$ since n depends on q. Further, extending the analysis of the mean hitting time $\mathbb{E}_{\omega^*}(\tau_x)$ to arbitrary vertices x of the form $x = n\mathbf{x}$, where $\mathbf{x}$ is unit vector of $\mathbb{R}^d_+$ and $n \in \mathbb{N}$, using capacity methods as in [13] seems prohibitive.

3.2 Main results

Let us now come to the contributions of this thesis.

3.2.1 Front velocity bounds

Theorem 3.6 served as a main inspiration for our first result in which we sought to get rid of the dependence of n on q and generalise the possible directions $\mathbf{x}$. Specifically, our main result concerns the small q behaviour of $v_{\max}(\mathbf{x}), v_{\min}(\mathbf{x})$ as a function of $\mathbf{x} \in \mathbb{R}_+^d$. We will distinguish between the case in which the direction $\mathbf{x}$ is fixed independent of q and the case in which $\mathbf{x} = \mathbf{x}(q)$ and $\min_i \mathbf{x}_i \to 0$ as $q \to 0$.

Theorem 1. *Fix $d \geq 2$.*

(A) Let $\mathbf{x} \in \mathbb{R}_+^d$ be a unit vector with $\min_i \mathbf{x}_i > 0$. Then

$$\lim_{q\to 0} -\frac{2}{\theta_q^2} \log_2(v_{\max}(\mathbf{x})) = \lim_{q\to 0} -\frac{2}{\theta_q^2} \log_2(v_{\min}(\mathbf{x})) = \frac{1}{d}.$$

(B) Let $0 < \beta < 1, \kappa \geq 1$ and let $\{\mathbf{x}(q)\}_{q\in(0,1)}$ be a family of unit vectors in $\mathbb{R}_+^d$ such that $\max_{i,j} \mathbf{x}_i(q)/\mathbf{x}_j(q) \leq \kappa 2^{\beta\theta_q}$. Then

$$1/d \leq \limsup_{q\to 0} -\frac{2}{\theta_q^2} \log_2(v_{\min}(\mathbf{x}(q))) < 1.$$

(C) Assume $d = 2$ and let $\alpha > 0$. Let $\{\mathbf{x}(q)\}_{q\in(0,1)}$ be a family of unit vectors in $\mathbb{R}_+^2$ such that $\max_{i,j} \mathbf{x}_i(q)/\mathbf{x}_j(q) \geq 2^{\alpha\theta_q^2}$. Then

$$\liminf_{q\to 0} -\frac{2}{\theta_q^2} \log_2(v_{\max}(\mathbf{x}(q))) \geq \frac{(1+4\alpha)\wedge 2}{2}.$$

Moreover, if $\alpha > 1/4$ then

$$\lim_{q\to 0} -\frac{2}{\theta_q^2} \log_2(v_{\max}(\mathbf{x}(q))) = \lim_{q\to 0} -\frac{2}{\theta_q^2} \log_2(v_{\min}(\mathbf{x}(q))) = 1.$$

The same results apply to $\hat{v}(\mathbf{x})$ from Section 3.1.1.

Remark 3.7. Part (C) is presented here only for $d = 2$ for simplicity. Remark 2.1 and the same proof ideas give similar, although more involved, results also for $d \geq 3$.

By combining (A) above together with Remark 2.1 we immediately get

Corollary 2. *Fix $d \geq 2$ and let $\mathbf{x} \in \mathbb{R}_+^d$ be a unit vector such that $\min_i \mathbf{x}_i = 0$. Then*

$$\lim_{q\to 0} -\frac{2}{\theta_q^2} \log_2(v_{\max}(\mathbf{x})) = \lim_{q\to 0} -\frac{2}{\theta_q^2} \log_2(v_{\min}(\mathbf{x})) = \frac{1}{d(\mathbf{x})},$$

where $d(\mathbf{x}) := \#\{i \in [d] :\ \mathbf{x}_i > 0\}$.

Remark 3.8. In order to better understand Theorem 1, let us write the full lattice $\mathbb{Z}^d$ spectral gap from Theorem 2.3 for $d \geq 1$ in the same form:

$$\lim_{q\to 0} -\frac{2}{\theta_q^2} \log_2(\gamma_d) = 1/d.$$

Notice that $\gamma_{d+1} = \gamma_d^{(1+o(1))d/(d+1)}$ and that this also holds for the East process on equilateral boxes with maximal boundary conditions by Theorem 2.3. Then:

(A) if the direction $\mathbf{x}$ points towards the *bulk* of $\mathbb{R}_+^d$ *uniformly in q as $q \to 0$*, then $v_{\max}(\mathbf{x}) = v_{\min}(\mathbf{x})^{1+o(1)} = \gamma_d^{1+o(1)}$;

(B) if $\mathbf{x} = \mathbf{x}(q)$ points to a lower dimensional space *slowly enough* as $q \to 0$, then $v_{\min}(\mathbf{x})$ is much larger than the velocity $v(\mathbf{e}), \mathbf{e} \in \mathcal{B}$, in any coordinate direction;

(C) for $d = 2$ if $\mathbf{x} = \mathbf{x}(q)$ approaches one of the coordinate directions *fast enough*, then $v_{\max}(\mathbf{x})$ is much smaller than the minimal velocity in the bulk and if this convergence is strictly fast enough, then $v_{\max}(\mathbf{x}) = v_{\min}(\mathbf{x})^{1+o(1)} = v(\mathbf{e}_1)^{1+o(1)}$.

3.2.2 Mixing set behind the front

The second result analyses the law at large times of the East process with initial condition ω^*, which we recall is the state with no vacancy so that at the start only the origin is unconstrained by assumption. It proves that for q small enough the region of $\mathbb{Z}_+^d$ where the East process at time t has relaxed to the reversible measure μ is extremely elongated in the bulk of $\mathbb{Z}_+^d$ (see Figure 1.1).

Theorem 3. *Fix $d \geq 2, 0 \leq \delta < 1$ and $\varepsilon > 0$. Let*

$$\Lambda(\delta, \varepsilon, t) = \{x \in \mathbb{Z}_+^d : \min_{i,j} x_i / x_j \geq \delta \text{ and } \|x\|_1 \leq 2^{-\frac{\theta_q^2}{2d}(1+\varepsilon)} \times t\}, \quad t > 0,$$

and let $\nu_t^{\delta,\varepsilon}$ be the marginal on $\Omega_{\Lambda(\delta,\varepsilon,t)}$ of the law of the East process at time t with initial condition ω^*. Then,

$$\limsup_{\varepsilon\to 0} \limsup_{q\to 0} \limsup_{t\to\infty} \|\nu_t^{\delta,\varepsilon} - \mu_{\Lambda(\delta,\varepsilon,t)}\|_{TV} = 0 \quad \text{if } \delta > 0, \tag{3.3}$$
$$\liminf_{\varepsilon\to 0} \liminf_{q\to 0} \liminf_{t\to\infty} \|\nu_t^{\delta,\varepsilon} - \mu_{\Lambda(\delta,\varepsilon,t)}\|_{TV} = 1 \quad \text{if } \delta = 0.$$

Remark 3.9. A slightly more refined formulation of Theorem 3 avoiding the $\limsup$ on ε, q would have been possible. However, we opted for the present version for simplicity.

3.2.3 Cutoff phenomenon

Let us come to our final result which is on the cutoff on $\Lambda_n = \{0, \dots, n\}^2$. Consider the time

$$T_n = n/v,$$

where v is the front velocity along any coordinate direction $\mathbf{e} \in \mathcal{B}$ which we recall is given by the velocity of the one dimensional East process (see Remark 3.1). Recall that $d_n(t) = \max_{\omega\in\Omega_{\Lambda_n}} \|\mathbb{P}_\omega^t(\cdot) - \mu_{\Lambda_n}\|_{TV}$, where $\mathbb{P}_\omega^t(\cdot)$ denotes the law at time t of the East process on Λ_n with initial condition ω.

Theorem 4. *There exists $q_0 \in (0, 1)$ such that for any $0 < q \leq q_0$*

$$\lim_{\alpha\to\infty} \liminf_{n\to+\infty} d_n(T_n - \alpha\sqrt{n}) = 1, \tag{3.4}$$
$$\limsup_{n\to+\infty} d_n(T_n + n^{2/3}) = 0. \tag{3.5}$$

Remark 3.10. Above we didn't try to optimise the cutoff window size. Using Theorem 3.5, T_n is the mixing time of the standard one dimensional East chain on the interval $\{0, \dots, n\}$. Hence, in a very precise sense, the one dimensional evolution along the coordinate axes dominates the mixing process of the multidimensional East chain in Λ_n.

Chapter 4

Multicolour East models

This chapter introduces the *multicolour East model* (MCEM). We start in Section 4.1 with a physical motivation for the definition and present some heuristics for the two colour case on $\mathbb{Z}^2$. Then, in Section 4.2 we give the formal construction showing the parallels to the East model construction from Chapter 2. Finally, in Section 4.3 we present a result giving positivity of the spectral gap in $\mathbb{Z}^d$ for specific choices of the geometry of the model. Further, we show sufficient conditions for the spectral gap of the three colour model to tend to γ_2 as the minimum equilibrium density goes to 0.

4.1 Introduction

We start with the conceptual shortcomings that the East model has, as a physical model for glass-forming liquids and which the MCEM tries to remedy. Then we come to ergodicity considerations on $\mathbb{Z}^d$ and show a first bound on the spectral gap in $\mathbb{Z}^2$ for the two colour model.

4.1.1 Physical motivation

Let us recall the physical motivation for analysing the East model and use that to motivate the definition of MCEM loosely following the arguments given in [27]. The modelling of supercooled glass-forming liquids by the East model relies on three assumptions.

The first is that most movement in the liquid is not through diffusion but vibration, so atoms do not wander through the lattice but get locally excited. The second is that atom mobility is facilitated and that most atoms are jammed. Facilitated in this case means that an atom can move if there is a neighbouring excited space (i.e. a vacancy). The assumption that most atoms are jammed, or alternatively that the temperature is low, is necessary because it is to be assumed that in the high-temperature region other dynamics are more prevalent than the glass-forming dynamics that lead to non-Arrhenius relaxation. The third assumption is that facilitated mobility carries a direction. If an atom vibrates in a way to create a gap in a south-western direction then the resulting gap should facilitate movement in the north-east neighbourhood of this atom.

The first assumption is modeled in the East model as a vacancy induces another vacancy rather than having a diffusive motion like the simple exclusion process through the lattice. The second assumption is modeled through the constraints of the East model where the assumption that most vertices are jammed is given if we take $q \sim e^{-\beta}$ small where β is the inverse temperature. The third assumption is again modeled in the constraints of the East model in which only the north-east neighbours of a vacancy are unjammed. A priori there is no reason to impose a unique direction of facilitation throughout a whole lattice, as it might be that an atom gets excited such that its north and west neighbours get unjammed, while somewhere else completely an excitation that unjams the south and east neighbours takes place. The MCEM seeks to remedy this by allowing for multiple vacancy types that each have a different associated direction to them of neighbours that they unjam.

We first present the model introduced by Chandler and Garrahan in [27], which we call the isotropic MCEM and then present the version used in this thesis, the MCEM. The isotropic MCEM is a two-parameter model with parameters $q, \xi \in [0, 1]$. On $\mathbb{Z}^d$ the state space consists of a neutral state $\star$ and 2^d vacancy types where each vacancy type unjams d neighbours. The set of facilitated neighbours is given by a rotation of the original d-dimensional East model constraints associated to each vacancy type, e.g. for $d = 2$ there is a vacancy type that respectively facilitates transitions in the respective north-east, north-west, south-east and south-west neighbourhood (corresponding to the four $\pi/2$ rotations of the original East constraints).

The isotropic MCEM dynamics only allow transitions from the neutral state $\star$ to a vacancy state with rate $q/2^d$ and from a vacancy state back to the neutral state with rate $1 - q$, but no transitions in between vacancy states of different types. There are two types of transitions governed by the parameter ξ. With rate ξ the vacancy-neutral state transitions happen if facilitated by a vacancy of one of the above 2^d vacancy types in the correct spot. With rate $1 - \xi$ we ignore interactions between the vacancy types. More precisely, if there is a vacancy of some type on x then only a vacancy of the same type on the appropriate spot can facilitate a transition to $\star$. Conversely if x is in state $\star$ it can only transition to those vacancy types that facilitate its transition.

We call the model isotropic MCEM since there is no preferred propagation direction inherent in the model. Physically the model with $\xi \in (0, 1)$ mirrors the fact that directionality in the facilitation dissipates on a time scale $1/\xi$. If $\xi = 0$ the model rigidly requires a chain in space-time on the lattice of equal vacancy types, if $\xi = 1$ no directionality is inherited throughout chains of facilitation. In fact, for $\xi = 1$ the dynamics resemble that of the FA-1f KCM since diffusive motion of vacancy types is possible if more involved than in the FA-1f case (see Figure 1 of [27]). For $\xi = 0$ it resembles that of multiple rotated versions of the East process evolving on the same lattice only sharing their particle state as the neutral state $\star$.

In this picture, the various East-models block each other since if there is a frequent vacancy type it might completely block the infrequent vacancy types from evolving on the lattice. In fact, this mutual blocking leads to the natural conjecture that the spectral gap for the $\xi \in \{0, 1\}$ cases is much lower than for their respective KCM counterparts. Contrary to this, in [27] it is conjectured, based on simulation, that for $\xi = 1$ the spectral gap does indeed scale similarly to the FA-1f model and for $\xi = 0$ similarly to the East model and for $\xi \in (0, 1)$ the spectral gap exhibits a scaling given by an interpolation between the scaling of the two edge cases. It seems to have been missed in [27] that for $\xi = 0$ the isotropic MCEM is not ergodic (see Theorem 5(A) further down).

The $\xi = 0$ case is the basis for the definition of the MCEM and is the subject for the second main problem dealt with in this thesis, and the first result identifies versions of it with less vacancy types that are exponentially ergodic, i.e. have positive spectral gap (Theorem 5(B)). In the MCEM we do not fix the rate of transition to the vacancy types to be uniform, hence our model is also anisotropic. Further, in Theorem 6 we consider the two-dimensional three vacancy type case and we show sufficient conditions on the equilibrium distribution for the spectral gap to tend to γ_2 as the equilibrium density of the least frequent vacancy type tends to 0. Unexpectedly, we also show this for cases where there are other frequent vacancy types.

4.1.2 Heuristic analysis of the two colour case

Consider first the MCEM on $\mathbb{Z}^1$ with a vacancy type that behaves like the regular East vacancy, call it A-vacancy, and another, call it B-vacancy, that behaves like a mirrored East vacancy, i.e. requiring that the right neighbour has a B-vacancy. This model is not ergodic. Indeed, let $q_A > 0$ and $q_B > 0$ be the appropriate equilibrium densities and consider a starting state in equilibrium. Almost surely there is a pair of vertices $x, x + \mathbf{e}_1$ such that there is a B-vacancy on x and an A-vacancy on $x + \mathbf{e}_1$. To remove the B-vacancy $x + \mathbf{e}_1$ needs to have a B-vacancy and vice-versa for $x + \mathbf{e}_1$ and A-vacancies. Thus, in the MCEM process as described above there is no legal transition for a pair like this, and thus this MCEM is not ergodic.

This is generalisable to $\mathbb{Z}^d$ and indeed the reason why the isotropic MCEM with $\xi = 0$ is not ergodic. Showing ergodicity is a more involved process requiring the cooperation of the various vacancy types and we state our results for this in Theorem 5.

Consider an MCEM on $\mathbb{Z}^2$ with the analogous two vacancy types, i.e. A-vacancies that have regular East model constraints and B-vacancies that are mirrored. Let $q_A = 1/2$ and $q_B = \varepsilon$ for $\varepsilon \ll 1$. We want to argue that the spectral gap is lower bounded by the spectral gap $\gamma_1(q_B)$ of the one-dimensional East model with vacancy density q_B. Indeed, we will see that it suffices to find a very likely event so that the origin relaxes, i.e. can transition to both the A-vacancy and B-vacancy state, on a time scale $\gamma_1^{-1}(q_B)$ given that event[1].

For small ε it is not hard to convince oneself that it is exponentially likely that there is a $x \prec 0$ not too far from the origin with an A-vacancy and such that on a shortest path from x to 0 there is no B-vacancy. For the B-vacancies this is not possible, since any B-vacancy is submersed in a sea of A-vacancies.

The question is how, despite this, a B-vacancy can travel to the origin in the time-window of a regular one-dimensional relaxation. Consider the first vertex with a B-vacancy in the $\mathbf{e}_2$ direction from the origin and call it ξ. We can consider any A-vacancy between 0 and ξ as a neutral state at little cost in relaxation time. Indeed, similarly to transitioning the origin to the A-vacancy state, any A-vacancy between 0 and ξ is exponentially likely to have a short down-left path to an A-vacancy on which there is no B-vacancy. The average waiting time to create A-vacancies or neutral states is roughly the expectation of a geometric variable and given by $1/q_A = 2$ and $1/(q_A + q_B) \leq 3$ respectively. Removing the A-vacancies between 0 and ξ thus has a constant cost. The dynamics of this models are thus given by the B-vacancy that starts to move down from ξ to 0 in a one-dimensional East fashion while occasionally having to wait for a constant time until an A-vacancy in the way gets removed.

Already for two-vacancy types and just showing one-dimensional East relaxation we see that some cooperation is necessary. In Theorem 6 we show sufficient conditions on the equilibrium distribution that the three vacancy type model has two-dimensional East relaxation. This requires complex constructions that show that cooperation is possible allowing the least frequent vacancy types to relax two-dimensionally while the frequent vacancy types eliminate each other.

4.2 Construction of the MCEM process

Some notation will be recycled since the construction of the MCEM will be analogous to the definition of the standard East model given in Chapter 2.

Definition 4.1 (Vacancy types and their constraints)**.** The set of vacancy types is a finite set $\mathcal{V}$ of cardinality 2^d. We identify $\mathcal{V}$ with the hypercube $H_d := \{0,1\}^d \subset \mathbb{Z}^d$ and refer to the vacancy type corresponding

[1] For those who know: this is done using the exterior condition theorem from [44], otherwise see Chapter 5 for some more details.

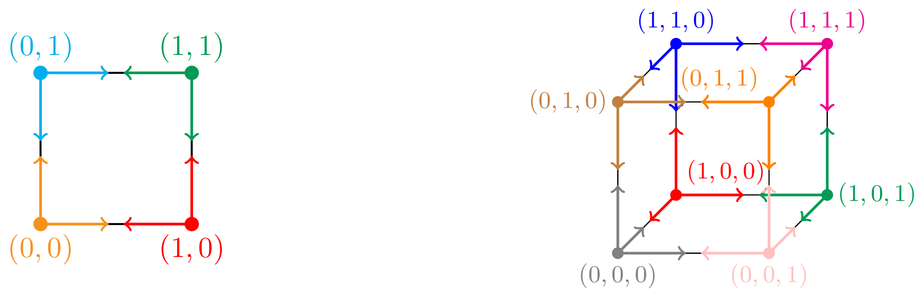

FIGURE 4.1 H_d for $d = 2$ (left) and $d = 3$ (right) together with the vacancy types as coloured corners and the propagation directions of the corners as arrows pointing away from the appropriate corner, the length of the propagation directions is less than half the actual length for rendering reasons.

to the vertex $h \in H_d$ as *the vacancy of type h or the h-vacancy*. We say that $\mathbf{v}$ is a *propagation direction* for the h-vacancy, and write $\mathbf{v} \in \mathcal{P}(h)$, if $\|\mathbf{v}\| = 1$ and $h + \mathbf{v} \in H_d$. For $h \in H_d$ we say that $x \prec^{(h)} y$ if $x \cdot \mathbf{v} \leq y \cdot \mathbf{v}$ for every $\mathbf{v} \in \mathcal{P}(h)$.

Given $G \subset H_d$ which we identify with a collection of vacancy types in $\mathcal{V}$, the single vertex state space $\mathcal{S}(G)$ will consist of G and a *neutral state* denoted by $\star$. For $\omega \in \mathcal{S}(G)^{\mathbb{Z}^d}$, $h \in G$ and $x \in \mathbb{Z}^d$ the constraint $c_x^h(\omega)$ is given by

$$c_x^h(\omega) = \begin{cases} 1 & \text{if } \exists\, \mathbf{v} \in \mathcal{P}(h):\ \omega_{x-\mathbf{v}} \text{ is a } h\text{-vacancy}, \\ 0 & \text{otherwise.} \end{cases}$$

We refer to Figure 4.1 for an illustration of H_2 and H_3 with the associated propagation directions for each vacancy type.

Remark 4.2. If $G = \{(0, 0, \dots, 0)\}$, we can identify $\star$ with 1 and $(0, 0, \dots, 0)$ with 0 to recover the state space of the d-dimensional East model with the corresponding constraints on $\mathbb{Z}^d$. Notice in particular the difference to the situation for the front evolution problem, discussed in Remark 2.4. Where for the front evolution problem it makes sense to consider the East model on $\mathbb{Z}_+^d$ and fix minimal boundary conditions so that the origin is always unconstrained, for the MCEM we consider $\mathbb{Z}^d$ as the base case.

Notation warning: In the sequel, for a given $\omega \in \mathcal{S}(G)^{\mathbb{Z}^d}$ and $h \in G$, we will often write $\omega_x = h$ meaning that ω_x is a vacancy of type h.

For $G \subset H_d$ we call vectors $\mathbf{q} = \{q_h \colon h \in G\}$ with $q_h > 0$ for $h \in G$, and $\sum_{h \in G} q_h < 1$, *valid parameter sets* and write $p = 1 - \sum_{h \in G} q_h$. Given a valid parameter set $\mathbf{q}$ let ν denote the probability measure on $\mathcal{S}(G)$ that assigns probability p to the state $\star$ and q_h to h for all $h \in G$. For any $\Lambda \subset \mathbb{Z}^d$ define the state space $\Omega_\Lambda = \mathcal{S}(G)^\Lambda$ and the measure $\mu_\Lambda := \otimes_{x \in \Lambda} \nu$, where we recall the notational convention that we leave away Λ if $\Lambda = \mathbb{Z}^d$. We also omit the dependence on $\mathbf{q}$ and G in the notation of p, ν and Ω_Λ as they will be clear from context.

Definition 4.3 (The G-MCEM process)**.** Given a subset $G \subset H_d$ and a valid parameter set $\mathbf{q}$ we define the continuous time G-MCEM process on $\mathbb{Z}^d$ via the infinitesimal generator, which we define through its action on local functions $f : \Omega \to \mathbb{R}$, as

$$\mathcal{L} f(\omega) = \sum_{h \in G} \sum_{x \in \mathbb{Z}^d} c_x^h(\omega) [\mathbb{1}_{\omega_x = \star} q_h + \mathbb{1}_{\omega_x = h} p] \nabla_x^{(h)} f(\omega),$$

where

$$\nabla_x^{(h)} f(\omega) := \begin{cases} f(h \cdot \omega_{\mathbb{Z}^d \setminus \{x\}}) - f(\omega) & : \text{if } \omega_x = \star, \\ f(\star \cdot \omega_{\mathbb{Z}^d \setminus \{x\}}) - f(\omega) & : \text{if } \omega_x = h, \\ 0 & : \text{else}. \end{cases}$$

We write $\omega(t)$ for the state at time t and $\mathbb{E}_\eta$ and $\mathbb{P}_\eta$ for the corresponding expectation and law for the process started at $\eta \in \Omega$.

Remark 4.4. It might be surprising that the sum of the rates $q_h + p < 1$ as opposed to usual KCM case in which no matter what state a vertex was in the rings came in with rate 1. In fact, here the missing rate $1 - q_h - p$ is hidden in $\nabla_x^{(h)} f(\omega) = 0$ if $\omega_x \notin \{h, \star\}$, thus we could have added a term $\mathbb{1}_{\omega_x \notin \{h,\star\}}(1 - q_h - p)$ for the ring that does nothing.

Remark 4.5. Notice that in the G-MCEM process a state can transition from $\star$ to h iff there is a vector $\mathbf{v} \in \mathcal{P}(h)$ such that $x - \mathbf{v}$ has an h-vacancy justifying the name *propagation direction* for $\mathbf{v}$. In particular, an h-vacancy at x can only influence those vertices y such that $x \prec^{(h)} y$. Further, there is no transition from one vacancy type to another. The process, in order to change the state of a vertex from one vacancy type to another, first has to go through the neutral state $\star$ (justifying its name). In particular, when $|G| \geq 2$ an h-vacancy can be blocked by a cluster of nearby vacancies of type in $G \setminus \{h\}$. This blocking interaction requires some new ideas w.r.t. the standard multidimensional East process in order to prove the main results below.

Reversibility in the case of KCM followed from the fact that the constraints did not depend on x, which is also the case here, but as opposed to KCM, for the transition term, $\nabla_x^{(h)} f(\omega)$, to not be zero requires $\omega_x \in \{\star, h\}$. Reversibility with respect to μ still follows (analogously to the KCM case) since for $\omega' \in \Omega_{\mathbb{Z}^d \setminus \{x\}}$ we have

$$\begin{aligned} \sum_{\omega \in \Omega_x} & \mu_x(\omega) f(\omega \cdot \omega')[\mathbb{1}_{\omega=\star} q_h + \mathbb{1}_{\omega=h} p] \nabla_x^{(h)} g(\omega \cdot \omega') \\ & = p q_h (f(\star \cdot \omega') - f(h \cdot \omega'))(g(h \cdot \omega') - g(\star \cdot \omega')), \end{aligned}$$

and thus

$$\mu(f \mathcal{L} g) = \mu(g \mathcal{L} f),$$

from which reversibility follows since f,g were arbitrary. The associated Dirichlet form is then

$$\begin{aligned} \mathcal{D}(f) := \mu(-f \mathcal{L} f) &= \sum_{h \in G} \sum_{x \in \mathbb{Z}^d} p q_h \mu \left[c_x^h (\nabla_x^{(h)} f)^2 \right] \\ &= \sum_{h \in G} \sum_{x \in \mathbb{Z}^d} p q_h \mu \left[c_x^h \mathbb{1}_{\omega_x \in \{\star, h\}} (f(\star \cdot \omega) - f(h \cdot \omega))^2 \right] \end{aligned} \tag{4.1}$$

and we define the spectral gap as

$$\gamma(G; \mathbf{q}) = \gamma(G) := \inf_{\substack{f \in \mathrm{Dom}(\mathcal{L}) \\ f \neq \text{const}}} \frac{\mathcal{D}(f)}{\mathrm{Var}(f)} . \tag{4.2}$$

The proof for Theorem 2.2 holds for general interacting particle system so that ergodicity of the MCEM process follows if 0 is a simple eigenvalue of $\mathcal{L}$ and in particular follows if the spectral gap is positive.

Remark 4.6. We will sometimes write $\mathcal{L}_\Lambda$ and $\mathcal{D}_\Lambda$ in which the sum $\sum_{x \in \mathbb{Z}^d}$ is replaced with a sum over $\sum_{x \in \Lambda}$ and the measure μ with the measure μ_Λ. The associated process on Λ is, in general, not ergodic due to the lack of specified boundary conditions.

Graphical construction

An alternative to the construction via the infinitesimal generator is again via a graphical construction. Put a marked Poisson process on each vertex in $\mathbb{Z}^d$. The k-th ring at the vertex $x \in \mathbb{Z}^d$ occurs at time $t_{x,k}$ and for each ring we have the mark $U_{x,k} \sim \mu$ so that $U_{x,k} \in \mathcal{S}(G)$ and $\{U_{x,k}\}_{x,k}$ is an i.i.d. family. Consider a starting state $\omega(0) \in \Omega$ and denote by $\omega(t)$ the state at time $t \in \mathbb{R}_+$. With $t_{x,k}-$ an infinitesimally smaller time than $t_{x,k}$, the graphcially constructed process evolves as follows:

(i) At $t_{x,k}$ we say that we have a $U_{x,k}$-legal ring if any of the following conditions is satisfied

  (a) $U_{x,k} = \star$ and there is an $h \in G$ such that $\omega_x(t_{x,k}-) = h$ and $c_x^h(\omega(t_{x,k}-)) = 1$, or

  (b) $U_{x,k} \neq \star$, $\omega_x(t_{x,k}-) = \star$ and $c_x^{U_{x,k}}(\omega(t_{x,k}-)) = 1$, or

  (c) $U_{x,k} = \omega_x(t_{x,k}-)$ and $c_x^{U_{x,k}}(\omega(t_{x,k}-)) = 1$ (i.e. nothing changes).

(ii) If $t_{x,k}$ is an $U_{x,k}$-legal ring, we set $\omega_x(t_{x,k})$ equal to $U_{x,k}$.

As for the East process, showing that this construction is well defined on $\mathbb{Z}^d$ and leads to the same process as the one constructed above through the infinitesimal generator is analogous to the calculation done in [36] for the North-East model so we omit it again.

We find reversibility again since a transition in the graphically constructed process from $\star$ to h or vice versa keeps the local state in $\{\star, h\}$ and does not change the neighbourhood configuration thus maintaining reversibility.

Remark 4.7. To model the isotropic MCEM heuristically introduced in Section 4.1 let $G = H_d$ and $q_h = q/2^d$ for all $h \in G$. We can add a mark $U_{x,k}^{(0)}$ which is uniform in $[0,1]$ and have an additional parameter $\xi \in [0,1]$. Before checking step (i) say that you have a diffusive ring on x at $t_{x,k}$ if $U_{x,k}^{(0)} \leq \xi$ and an exclusive one otherwise. The exclusive one behaves as detailed above. For the diffusive ring step (i) is modified as follows:

(i) At $t_{x,k}$ we say that we have a $U_{x,k}$-legal ring if there is an $h \in G$ with $c_x^h(\omega(t_{x,k})) = 1$ and

  (a) $U_{x,k} = \omega_x(t_{x,k})$ or,

  (b) $U_{x,k} = \star$ and $\omega_x(t_{x,k}) \neq \star$ or,

  (c) $U_{x,k} \neq \star$ and $\omega_x(t_{x,k}) = \star$.

4.3 Main results

We now come to the main results.

4.3.1 Conditions for non-ergodicity and positivity of the spectral gap

The first result shows that the isotropic MCEM from Section 4.1 is not ergodic if $\xi = 0$ and gives sufficient conditions on G for the G-MCEM to be ergodic. Recall for this that any $G \subset H_d$ inherits the graph structure of $\mathbb{Z}^d$.

Theorem 5. *Consider all the following G-MCEM with an arbitrary valid parameter set* $\mathbf{q}$.

(A) If $G = H_d$ *then the G-MCEM process is not ergodic.*

(B) Suppose $G \subsetneq H_d$ *is such that either condition holds:*

(B.i) there is an $\mathbf{e} \in \mathcal{B}$ *such that for any two* $h, h' \in G$ *we have* $h \cdot \mathbf{e} = h' \cdot \mathbf{e}$*.*

(B.ii) there is a superset $G' \subsetneq H_d$ *of* G *such that* G' *is isomorphic to a star-graph.*

Then the G-MCEM process has a positive spectral gap.

Example 4.8. Any $G \subset H_3$ that is a subset of a single face satisfies (B.i) and any $G \subset H_2$ with $|G| < 4$ satisfies (B.ii). In particular note that this gives complete information about ergodicity in $d = 2$ but leaves gaps for $d \geq 3$.

4.3.2 Asymptotics as $q_{\min} \to 0$ of the spectral gap in $d = 2$

In Theorem 5 we establish the positivity of the spectral gap for specific choices of G, in fact for $d = 2$ we can do better than that. Given a valid parameter set $\mathbf{q}$ we define $q_{\min} = \min_{h \in G} q_h$, $q_{\max} = \max_{h \in G} q_h$ and if $|G| = 3$ we write q_{med} for the $q_h \in G$ that is not in $\{q_{\max}, q_{\min}\}$. If there are multiple q_h with $q_h = q_{\max}$ or $q_h = q_{\min}$ w.l.o.g. make a choice for which to call q_{med}. We further define $\theta_h = \theta_{q_h} := |\log_2(q_h)|$ and we recall that $\gamma_d = \gamma_d(q)$ denotes the spectral gap of the d-dimensional East model with vacancy density q and $\gamma(G; \mathbf{q})$ the spectral gap of the G-MCEM.

Theorem 6. *Fix* $\Delta > 0$ *and consider a G-MCEM on* $\mathbb{Z}^2$ *with* $|G| \in \{2, 3\}$ *and a valid parameter set* $\mathbf{q}$ *such that* $p > \Delta$*. Then,*

$$\lim_{q_{\min} \to 0} \frac{\gamma(G; \mathbf{q})}{\gamma_2(q_{\min})} = 1$$

in the following cases.

- *Any* 2*-subset* G *and either one of the following conditions holds:*
  - *(2.i)* $\lim_{q_{\min} \to 0} q_{\max} \theta^3_{q_{\min}} = 0$*,*
  - *(2.ii)* $\lim_{q_{\min} \to 0} q_{\max} \theta^3_{q_{\min}} / \log_2(\theta_{q_{\min}}) = \infty$*.*
- *Any* 3*-subset* $G \subset H_3$ *and either one of the following conditions holds:*
  - *(3.i)* $\lim_{q_{\min} \to 0} q_{\max} \theta^3_{q_{\min}} = 0$*,*
  - *(3.ii)* $\lim_{q_{\min} \to 0} q_{\max} \theta^3_{q_{\text{med}}} / \log_2(\theta_{q_{\min}}) = \infty$ *and* $\lim_{q_{\min} \to 0} q_{\text{med}} \theta^6_{q_{\min}} = 0$*,*
  - *(3.iii) G is such that the vacancies associated to* q_{med} *and* $q_{\max}$ *share a propagation direction and* $\liminf_{q_{\min} \to 0} q_{\text{med}} > 0$*.*

Remark 4.9. We exclude the one vacancy type case because it coincides with the (possibly rotated version) standard two-dimensional East model on $\mathbb{Z}^2$.

Heuristic considerations on Theorem 6 The cases are ordered from the easiest to the hardest regime. The first cases are the easiest since in these cases even the highest density $q_{\max}$ is relatively low so that most vacancies in equilibrium (from which the starting state is sampled) are surrounded by large neutral state patches. Thus for these cases it is natural to conjecture the conclusion of Theorem 6.

The next harder case is if there is one vacancy type that is frequent in equilibrium. Recalling the heuristic considerations on the AB-model in Section 4.1 the conclusion of Theorem 6 still presents itself as a natural conjecture if we consider that any vacancy of the frequent type will see large patches of either neutral vertices or its own vacancy type. Thus, any vacancy of the frequent type that blocks the infrequent vacancies is likely to be removable by close vacancies of the same type allowing the infrequent vacancies to evolve according to their respective two-dimensional East model dynamics.

The hardest case is for $|G| = 3$ when two vacancy types are frequent. In this case the frequent vacancy types might block each other and we only manage to find configurations that remove the blocking frequent vacancies if they share a propagation direction.

We end the section with a remark on the requirement $p > \Delta$ in Theorem 6. As just described our scheme to lower bound the spectral gap relies on finding configurations such that any interfering vacancies can be removed so as to allow the vacancies with the lowest density to relax in a two-dimensional way. Thus, these schemes imply that we need to wait for vacancies to become neutral, which means a contribution of order at least p to the spectral gap. In the limit $p \to 0$ we thus expect the spectral gap to go to 0 as well, and in fact, depending on the case we get a contribution to the lower bound of p or p^2 (see Remark 8.6). Since this neither constitutes the physically most interesting regime nor do we have matching upper bounds we decided to focus the presentation on the simpler case with $p > \Delta$.

Chapter 5

High-level overview of the main techniques

In this chapter we give an overview of the main ideas behind the proofs in this thesis, hopefully giving an idea why they should be true. The presentation will naturally lead to an underlying principal Dirichlet eigenvalue problem that is shared for both the proof of Theorems 1 and 6 and is solved in Chapter 6.

5.1 Front evolution problem

We present the proofs of the theorems in order.

Front velocity bounds (Theorem 1)

Part (A)

Fix a unit vector $\mathbf{x} \in \mathbb{R}_+^d$ independent of q with $\min_i \mathbf{x}_i > 0$.

Lower bound on $v_{\min}(\mathbf{x})$ We wish to upper bound the average hitting time of $n\mathbf{x}$. We use an iterative argument and start by defining $x^{(n)} = \lfloor n\ell\mathbf{x} \rfloor$ for some $\ell = \ell(q)$. Since $\mathbf{x}$ is fixed, $x^{(\lfloor n/\ell \rfloor)}$ is not too far from $n\mathbf{x}$ and using the strong Markov property we find roughly that

$$\mathbb{E}_{\omega^*}(\tau_{n\mathbf{x}}) \geq n \max_{\omega \in \{\omega :\, \omega_{x^{(i-1)}} = 0\}} \mathbb{E}_\omega(\tau_{x^{(i)}}),$$

for some i. Thus, we need to find a good upper bound on the hitting time of a general $\tau_x^{(i)}$ starting with a vacancy in $x^{(i-1)}$ without specifying the state outside $x^{(i-1)}$. We choose ℓ such that with relatively little cost we can consider the equilibrium problem (cf. Equation (7.5)), i.e. where on the smallest box Λ containing $x^{(i-1)} + \mathbf{e}_1$ and $x^{(i)}$ there is equilibrium and the boundary condition otherwise is unknown. Notice that Λ is not equilateral but 'almost' equilateral, a condition we call $(0, \kappa)$-squeezed (cf. Definition 6.3). It is a small technical exercise (cf. Lemma 6.1) from this point to find that this equilibrium problem is upper bounded by $e^{-\lambda^D(\Lambda)t}$ where $\lambda^D(\Lambda)$ is the smallest eigenvalue for the Dirichlet problem

$$-\mathcal{L}_\Lambda f = \lambda f, \quad f \restriction_{\{\omega :\, \omega_{x_\Lambda} = 0\}} = 0.$$

Notice that the generator here has minimal boundary conditions, which we indicate by leaving away the superscript σ for the boundary condition.

Crucial in lower bounding $\lambda^D(\Lambda)$ for almost square sets is its connection to the spectral gap (see Equation (6.3)):

$$\lambda^D(\Lambda) \geq q \max\{\gamma(V) :\, V \subseteq \Lambda,\, V \supset \{0, x_\Lambda\}\}, \tag{5.1}$$

so the bound on $\lambda^D(\Lambda)$ follows by identifying a subset of Λ that has a spectral gap like γ_d. Using the renormalisation techniques first presented in [13] we iteratively find such a set V in Chapter 6 that, as one might expect, is concentrated around the main diagonal of Λ.

Upper bound on $v_{\max}(\mathbf{x})$ The goal is to find a lower bound on the time it takes to put a vacancy on a vertex $n\mathbf{x}$ when starting from the state ω^* with no vacancies. To that end we construct triangles $\Lambda_y = \{z \in \mathbb{Z}_+^d : z \prec y, \|y - z\|_1 \le \ell\}$ for some $\ell = \ell(q)$ such that y is its unique largest point in the $\prec$-order. By definition of the East model we have that $\tau_y \ge \tau_{\partial_\downarrow^+ \Lambda_y}$. Iteratively construct a sequence $\{\xi^{(i)}\}_i$ with $\xi^{(0)} = n\mathbf{x}$ and $\xi^{(i)} = \{y \in \partial_\downarrow^+ \Lambda_{\xi^{(i-1)}} : \tau_{\partial_\downarrow^+ \Lambda_{\xi^{(i-1)}}} = \tau_y\}$ (see Figure 5.1).

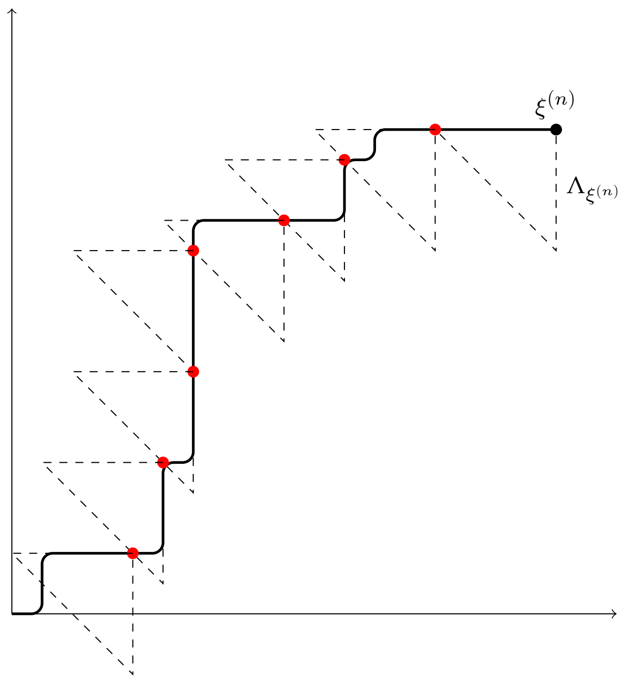

FIGURE 5.1 Example trajectory a vacancy could take in the upper bound on $v_{\max}(\mathbf{x})$ for part (A). For simplicity we just draw a path instead of the more correct two-dimensional shape. In dashed we have the triangles Λ_y together with the red points where the trajectory first hits $\partial_\downarrow^+ \Lambda_y$.

The side length $\ell(q)$ of the triangles is chosen independent of n so that we get $\Theta(n)$ many of them. Using the exponential Chebyshev inequality Equation (7.9) we can then reduce the problem of bounding $\mathbb{E}_{\omega^*}(\tau_{n\mathbf{x}})/n$ to the problem of $\Theta(n)$ times lower bounding the propagation speed on a single triangle, where the n then cancels. More precisely, on a triangle Λ_y with side length $\ell(q)$ and top right corner y we want to show that $\max_{\omega:\, \omega_{\Lambda_y} \equiv 1} \mathbb{P}_\omega(\tau_y < t) \to 0$ if $t = o(2^{\frac{\theta_q^2}{2d}})$ (see $W(\lambda)$ in Equation (7.9)).

We do this using a bottleneck. Indeed, in [13, Section 4] it is shown that there is an event A on Λ_y with $\mu(A) \le 2^{-\frac{\theta_q^2}{2d}(1+\varepsilon)}$ and $\tau_A < \tau_y$. Using this bottleneck the bound follows after some small calculations.

Part (B)

Repeat the same proof steps for the lower bound on $v_{\min}$ as in part (A). The main difference is that the set between $x^{(i)}$ and $x^{(i+1)}$ in the lower bound is not almost square but decidedly more rectangular than a line, we call it (β, κ)-squeezed with $0 < \beta < 1$. The corresponding upper bound for $\lambda^D(\Lambda)$ for such Λ is given in Proposition 6.6(ii) and is based on the bound in Proposition 6.6(i) through an interpolating renormalisation between the result for almost equilateral boxes and the one-dimensional case.

Part (C)

If $\mathbf{x}$ is such that $\max_{i,j} \mathbf{x}_i(q)/\mathbf{x}_j(q) \geq 2^{\alpha\theta_q^2}$, then for small q this means that $n\mathbf{x}$ is pretty close to an axis (recall that $d = 2$). Intuitively, we should find large stretches in which the vacancies that come to put a vacancy on $n\mathbf{x}$ evolve completely one-dimensionally.

To quantify this, we define a set U_y as in Figure 5.2 (which is the actual set we use from Definition 7.8). As before, starting at $n\mathbf{x}$ we consider the first vertex $\xi^{(1)} \in \partial_\downarrow^+ U_{n\mathbf{x}}$ that had a vacancy and then consider the first vertex $\xi^{(2)} \in \partial_\downarrow^+ U_{\xi^{(1)}}$ that had a vacancy and so on until this sequence reaches the origin (cf. *infection sequence* Definition 7.9). We thus can find the desired bound by considering the infection time of $\xi^{(i)}$ started from a state with the only vacancy on $U_{\xi^{(i)}}$ being $\xi^{(i+1)}$.

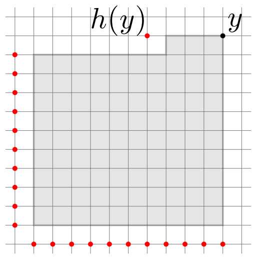

FIGURE 5.2 Example for a set U_y (the grey region). The red vertices denote $\partial_\downarrow^+ U_y$. For rendering purposes the distance between $h(y)$ and y is drawn comparable in size to the distance of $h(y)$ from $\partial_\downarrow^+ U_y \setminus \{h(y)\}$, when in reality the former is much smaller (see text).

We distinguish two cases: the case when $\xi^{(i+1)} = h(\xi^{(i)})$ and the case when not. The former case is the interesting one since, if the first vacancy in $\partial_\downarrow^+ U_y$ is $h(y)$ then, no matter what the boundary condition outside of U_y, even the diagonal mode cannot influence the one-dimensional vacancy propagation that goes from $h(y)$ to y. This is because the boundary $\partial_\downarrow^+ U_y \setminus \{h(y)\}$ is chosen far enough away from the linear strip that connects $h(y)$ to y.

Thus, the propagation speed from $h(y)$ to y is given by the one-dimensional East process and we can use the comparatively tighter bottleneck for the one-dimensional process as opposed to the d-dimensional one. Finally we combinatorially prove that the number of i such that $h(\xi^{(i)}) = \xi^{(i+1)}$ is high enough for the lower bound to follow (Lemma 7.11).

Mixing behind the front (Theorem 3)

Consider Figure 5.3 in which we zoomed into the lower left 100×100 vertices of the simulation results from Figure 1.1. We roughly added the area covered by $\Lambda(1/2, \varepsilon, t')$ for some $t' < t$. The closer δ gets to 0 the closer the two longest sides run along the axes. Knowing that the mixing time on the equilateral box of side length n is $\Theta(n)$ and now knowing that the propagation speed in any q-independent direction tends to γ_d it should not be surprising that we relax to equilibrium in $\Lambda(\delta, \varepsilon, t) \subset \mathbb{Z}_+^d$ if it does not include the axes, i.e. $\delta > 0$ and do not if it includes the axes, i.e. $\delta = 0$. In fact, the proof follows exactly this intuition by relating the total variation at time t with the probability of hitting any vertex in $\Lambda(\delta, \varepsilon, t)$ in less than $t/3$ which then leaves enough time to reach equilibrium (cf. Equation (7.14)).

As discussed above, the bound in Theorem 1(A) depends on Proposition 6.6(i) which in turn depends on the original result of Theorem 2.3 given in [13, Theorem 2] which was fairly precise as it also included lower order corrections. Here we only consider the highest order so taking $t \to \infty$ is crucial to ensure that $\Lambda(\delta, \varepsilon, t) \subset C(t)$.

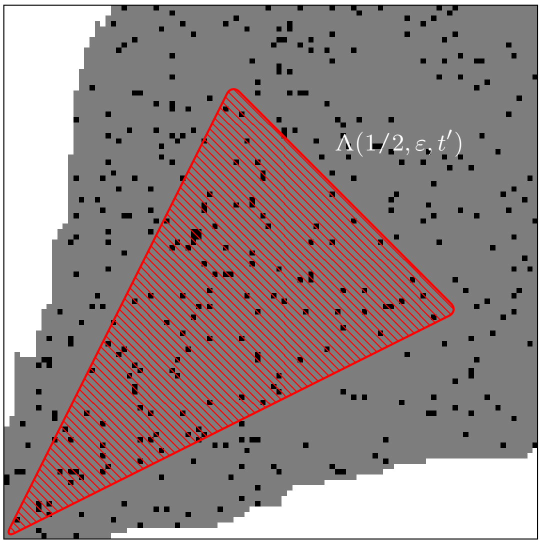

FIGURE 5.3 Cropped lowest 100×100 part of Figure 1.1. In red we drew (approximately) the vertices included in $\Lambda(1/2, \varepsilon, t')$ for appropriate ε and t'. **Attention:** For the time in this particular simulation, the t' relative to the set $\Lambda(1/2, \varepsilon, t')$ and the time t for the process are not the same, usually for small t we have $C(t) \subset \Lambda(\delta, \varepsilon, t)$ where we recall that $C(t)$ is the set of vertices that had a ring before t (i.e. the grey and black ones). We draw the image here to give an idea of the shape of $\Lambda(\delta, \varepsilon, t')$.

Cutoff on the box (Theorem 4)

Recall from Theorem 3.5 that the mixing time $T^{(n)}_{\text{mix}}$ on $\Lambda_n = \{0, 1, \dots, n\}$ exhibits a cutoff with window $\sqrt{n}$. We exploit the geometry of the boxes Λ_n together with the chosen boundary conditions for the East chain, which we recall is minimal, and again use the fact that for small q the front velocity along the coordinate axes is much smaller than the minimal velocity in any other direction pointing towards the bulk of Λ_n. This implies that the part of Λ_n not on the axes is already mixed long before every vertex on the axis had a legal ring, so that the mixing time is governed by the one-dimensional mode for which we know that there is cutoff. Indeed, this intuition is supported by Figure 5.3 in which we see large parts of Λ_n already in $C(t)$ while less than a quarter of $\{0, \dots, 99\} \cdot \mathbf{e}$ for $\mathbf{e} \in \mathcal{B}$ is in $C(t)$.

5.2 MCEM

Ergodicity and positivity of the spectral gap (Theorems 5 and 6)

Both results for the MCEM rely on the *exterior condition theorem* [44, Theorem 2] that we recall in Theorem 8.2. Roughly, it says that if we find a family of high probability events $\{\mathcal{A}_x\}_{x \in \mathbb{Z}^d}$ that only depend on vertices 'on one side of' x (the exterior condition) then

$$\mathrm{Var}(f) \leq 4 \sum_{x \in \mathbb{Z}^d} \mu(\mathbb{1}_{\mathcal{A}_x} \mathrm{Var}_x(f)). \tag{5.2}$$

The goal is then to find such events that let us upper bound the r.h.s. by the Dirichlet form of the G-MCEM while only paying with an appropriate term depending on $\mathbf{q}$. This term then is the lower bound on the spectral gap. We do this by identifying configurations that allow us to define legal paths to transition x between any two states (recall the discussion of the AB-model in Section 4.1). Let us thus describe the main ideas behind finding the legal paths, the detailed analysis of how to go from a legal path to a lower bound on the spectral gap is left to the proofs.

In the proof of Theorem 2.2 we do not care about the nature of this term as long as it strictly positive for any $\mathbf{q}$ since we only want to prove the positivity of the spectral gap. Thus, we find configurations of vacancy types on boxes Λ together with some high-probability conditions on the environment of Λ such that we can clear any vacancy from $\Lambda + k\mathbf{v}$ for some direction $\mathbf{v}$ for any k. Looking far enough w.h.p. we find an m such that $\Lambda - m\mathbf{v} + x$ has this good configuration. Reversibility tells us that if we can remove any configuration from $\Lambda + k\mathbf{v}$ then we can also put any configuration so that we get the sought after legal path. The nature of these configurations is quite different for the cases when all vacancies share a direction (Theorem 5(B.i)) and when G is a star graph (Theorem 5(B.ii)) so we leave it at this point.

Spectral gap bounds in two dimensions (Theorem 6)

The proof of the various cases in Theorem 6 has two parts. First we need to argue that

$$\lim_{q_{\min} \to 0} \gamma(G, \mathbf{q}) / \gamma_2(q_{\min}) \leq 1.$$

Behind this is the intuitive notion that, the more vacancy types you introduce into your model the smaller the spectral gap should be. The spectral gap of the two-dimensional East process, being an MCEM with a single vacancy type, should thus upper bound $\gamma(G, \mathbf{q})$ (see Lemma 8.4).

This *monotonicity in* G of the spectral gap also means that for the upper bound it suffices to prove the cases $(3.x)$, as they then imply the corresponding cases $(2.x)$. Let us first discuss the case $(3.i)$ in which, we recall, $q_{\max}\theta^3_{q_{\min}} \to 0$ as $q_{\min} \to 0$. For simplicity we give the two-dimensional vacancy types names and call $A = (0, 0)$, $B = (1, 1)$, $C = (0, 1)$, $D = (1, 0)$ in agreement with the A- and B-vacancy type introduced in Section 4.1.2.

The mathematical tools we use are the same as for the proof of the positivity of the spectral gap above. The difficulty lies in trying to keep proportionality term between the r.h.s. of Equation (5.2) and the Dirichlet form of the order $O(\gamma_2(q_{\min}))$ for small $q_{\min}$.

(3.i) Let us discuss the salient ideas behind the proof of (3.i) by discussing the AD-model in the case (2.i), i.e. $q_{\min}\theta^3_{q_{\max}} \to 0$ as $q_{\min} \to 0$. In particular let us discuss the salient notions for the construction of the event on which we can use the exterior condition and which shows that the relaxation of A-vacancies follows chiefly two-dimensional East dynamics (see Figure 5.4).

First notice that A-vacancies propagate to the north and to the east. We thus start by identifying a set of $\Theta(q_A^{-1/2})$ north-east paths in the negative (or third) quadrant that start on the horizontal axis at a pairwise distance of $\Theta(\theta_A^{3/2})$, do not pairwise intersect and have a length of $\Theta(q_A^{-1/2})$ (see non-transparent paths in Figure 5.4). We call these the *vertical paths* and we then identify paths with the same parameters starting from the vertical axis and call them *horizontal paths*. Together they form the *grid* (see Definition 9.8).

Constructed as such, every horizontal path intersects each vertical path and so there is a graph isomorphism Φ that maps the last intersection points of the paths to an equilateral square (see Definition 9.9).

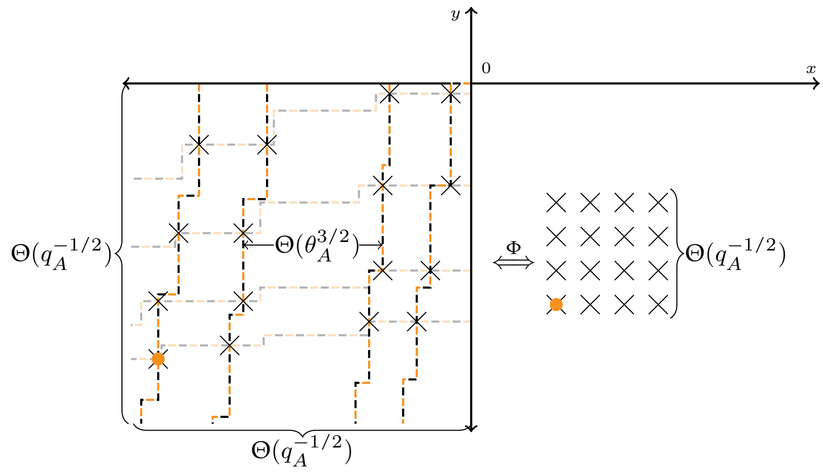

FIGURE 5.4 The construction for the A-vacancies (in orange here) in the proof of part (2.i) for the AD-model. The striped paths are the (random) paths on which there is no D-vacancy. We call the slightly transparent ones the horizontal and the others the vertical paths. Under the isomorphism Φ select intersection points of these paths are isomorphic to a $2D$-square.

W.h.p using a Peierls-type argument we find such a grid such that there is no D-vacancy on the entire grid. Further since the side length of the square given by Φ has side length $\Theta(q_A^{-1/2})$ w.h.p. there is an A-vacancy on the lower-left half of this square, so that we can use the exterior condition on the event that there is a grid with no D-vacancies and at least one A-vacancy on the intersection points.

We then consider an auxiliary model (Lemma 9.17) on the intersection points with the two-dimensional East constraints concatenated with Φ^{-1} where the A-vacancies play the role of vacancies and $\star$ the role of the particles (possible since we have no D-vacancies on the grid).

By the assumption that there is an A-vacancy in the lower left half, this auxiliary model has minimal boundary conditions. We again resort back to Proposition 6.6(i) and in particular use that we find subsets of boxes on which the East chain with minimal boundary conditions has the maximal spectral gap (recall the discussion around Equation (5.1)). In our case this means that we find a subset V of the intersection points such that the auxiliary model has spectral gap $\gamma_2(q_A)$.

To recover a Dirichlet form of the AD-model we finally need to propagate the A-vacancies between intersection points. We chose their pairwise distance low enough so that we can just take the one-dimensional East model on the paths between the intersection points and still have the two-dimensional dynamics on the intersection points dominate.

This brings the A-vacancy to the closest intersection point on the origin but not quite to the origin, so a simple extra construction is needed doing this last step.

Of course, we cannot use the exterior condition theorem just for the A-vacancies so we repeat the same construction with the mirrored events for the D-vacancies in the second quadrant and get part (2.i) for the AD-model.

This is the big picture idea for every configuration in part (2.i) and (3.i). There is a slight complication in the case of the AB-model (or a G-MCEM with $|G| = 3$ and $\{A, B\} \subset G$) since we need to do the construction for both A- and B-vacancies simultaneously. This is a problem since their dynamics are opposite, so that we need to do the construction in the positive and simultaneously in the negative quadrant while remaining 'on one side' of the origin as required by the exterior condition theorem. We solve this by always staying 'above' the main diagonal and adapting the construction correspondingly (see Figure 9.2).

The definition of a grid thus is more complex but the salient ideas are the same as the ones presented above.

(3.ii) Consider again the AD-model and assume that $q_D\theta_{q_A}^3 / \log_2(\theta_{q_A}) \to \infty$ as $q_A \to 0$. The goal is again to get a grid as in Figure 5.4, but this time we cannot find long paths that do not contain D-vacancies. Instead we consider the renormalised lattice in the negative quadrant of boxes of side length $L = L(q_A)$. We identify three possible configurations in these boxes that we call A-traversable, A-super and A-evil (see Definition 9.19).

The property of A-super boxes is that they contain a A-vacancy and on north-east paths of A-traversable boxes the A-super boxes can propagate like A-vacancies, i.e. if x is an A-traversable box and there is a $\mathbf{e} \in \mathcal{B}$ such that $x - \mathbf{e}$ is A-super then there is a legal path that makes x A-super.

We choose $L(q_A)$ such that A-traversability has increasing probability in decreasing q_A so that we can apply the construction from the case $(3.i)$ in the negative quadrant on the renormalised lattice for the auxiliary model where A-super vertices play the role of the A-vacancies, A-traversable vertices the role of the neutral state and A-evil vertices the role of D-vacancies (Corollary 9.22). The probability of being A-super is given in highest order by q_A so that we recover the spectral gap bound $\gamma_2(q_A)$.

The proof ends by showing that the two-dimensional East dynamics on the renormalised lattice dominates the spectral gap contribution gotten from considering the dynamics inside the boxes that propagate the A-super state on paths of A-traversable boxes (Lemma 9.26) and the adapted construction around the origin (Lemma 9.27).

(3.iii) This case is particular to the case where we have three-vacancy types so let us consider the ABC-model. Assume that $q_B = q_{\min}$ so that $\liminf_{q_B \to 0} q_A > 0$ and $\liminf_{q_B \to 0} q_C > 0$. To satisfy the exterior condition, that we recall, requires that the event $\mathcal{A}_x$ remains on 'one side of' x we do not need to take the diagonal anymore, since A-vacancies are very frequent so w.h.p we find them on a path that goes mostly left from x.

The main idea is still the same: We want to identify configurations on paths that allow a B-vacancy to travel horizontally and vertically and such that the relaxation time for the horizontal and vertical transitions is much smaller than the $\gamma_2(q_B)$ resulting from the grid of intersection points of these paths.

To do this, recall that A-vacancies propagate north and east while C-vacancies propagate south and east. This means that if north of a vertex x there is a C-vacancy and south there is an A-vacancy, then we can remove any non-B-vacancy from x, and in particular, if there was no B on x we can also remove any non-B-vacancy from $x + \mathbf{e}_1$, since $\mathbf{e}_1$ is the propagation direction A- and C-vacancies share. This means that in the above situation we can clear a horizontal line of vertices until we meet the first B-vacancy. This B-vacancy can then travel without hindrance since any A- or C-vacancies to its left have been removed. This is the horizontal propagation scheme we use.

The vertical case is analogous: in that we use that A and C vacancies are very frequent and remove any other non-B-vacancies in the $\mathbf{e}_1$ direction clearing the way for $-\mathbf{e}_2$ propagation of B-vacancies.

Notice in particular the difference to the first two cases, that here we find a grid of straight paths. Further, there are some intricacies here w.r.t. the exact conditions that we put onto the paths but this is the main idea to recover $\gamma_2(q_B)$ as the dominant term in the spectral gap.

Part II

Technical results and proofs

Chapter 6

Asymptotics of Dirichlet Eigenvalues via coarse graining and renormalisation group methods

This chapter gives the centerpiece technical result of this thesis which enters the proofs of Theorems 1, 3 and 6. It gives an improved solution on a suitable Dirichlet eigenvalues problem on boxes, w.r.t. the eigenvalue obtained by taking the spectral gap of the full box. We do this by using the Knight lattice and RG-group procedure introduced in [13] iteratively finding subsets of Λ with improved spectral gap bounds. We start in Section 6.1 by presenting the main result, Proposition 6.6, which contains three parts that are proved in order in Sections 6.2 to 6.4. We end the section by presenting Proposition 6.20, a slightly more nuanced version of Proposition 6.6.

6.1 The Dirichlet eigenvalue problem and its solution

It is a small technical exercise to link the hitting times of a specific event starting from equilibrium to the generator. Recall the convention for the East process on $\mathbb{Z}_+^d$ that if either σ is absent because $\partial_\downarrow^+\Lambda = \emptyset$ or $\sigma \equiv 1$, then the superscript σ is dropped from the notation and that in the East process the origin is always unconstrained.

Lemma 6.1 (See e.g. [3, Section 6]). *Let $\Lambda \subset \mathbb{Z}^2$ be a finite subset, $A \subset \Omega_\Lambda$ an event on Λ and denote by $\mathbb{P}$ and $\mathbb{E}$ the laws of a Markov chain on Λ with generator $\mathcal{L}$ and equilibrium measure μ. For the hitting time τ of A^c (note: the complement, not A) we find*

$$\sum_{\eta \in A} \mu(\eta)\mathbb{P}_\eta(\tau > t) = \langle \mathbf{1}, e^{t\mathcal{L}^D}\mathbf{1}\rangle,$$

where $t \in \mathbb{R}_+$, the scalar product is with respect to $\ell^2(\Omega_\Lambda, \mu)$ and $\mathcal{L}^D = \mathbf{1}_A\mathcal{L}\mathbf{1}_A$ is the generator with boundary conditions given by A.

Proof. Recall that

$$e^{t\mathcal{L}^D} f(\omega(0)) = \mathbb{E}_\eta(f(\omega(t))),$$

and

$$e^{t\mathcal{L}^D} = \sum_{m=0}^{\infty} \frac{(t\mathbf{1}_A\mathcal{L}\mathbf{1}_A)^m}{m!} = \mathbf{1}_A e^{t\mathcal{L}}\mathbf{1}_A,$$

and note that in our notation the state $\omega(s)$ of the chain at time s is always given by the immediate surrounding measure $\mathbb{E}_\eta/\mathbb{P}_\eta$ (so the time is relative to η and there can be multiple ω referring to different starting times). Using these properties we find for $s,t \in \mathbb{R}_+$

$$\begin{aligned}
e^{(t+s)\mathcal{L}^D} f(\eta) &= \Big[\mathbf{1}_A e^{t\mathcal{L}} \mathbf{1}_A e^{s\mathcal{L}} \mathbf{1}_A f\Big](\eta) \\
&= \mathbf{1}_{\eta\in A} \mathbb{E}_\eta\Big(\Big[\mathbf{1}_A e^{s\mathcal{L}} \mathbf{1}_A f\Big](\omega(t))\Big) \\
&= \mathbf{1}_{\eta\in A} \mathbb{E}_\eta\Big(\mathbf{1}_{\omega(t)\in A} \mathbb{E}_{\omega(t)}\Big(\mathbf{1}_{\omega(s)\in A} f(\omega(s))\Big)\Big) \\
&= \mathbf{1}_{\eta\in A} \mathbb{E}_\eta\Big(\mathbb{E}_{\omega(t)}\Big(\mathbf{1}_{\omega(0),\omega(s)\in A} f(\omega(s))\Big)\Big) \\
&= \mathbf{1}_{\eta\in A} \mathbb{E}_\eta\Big(\mathbb{E}_{\eta}\Big(\mathbf{1}_{\omega(t),\omega(s+t)\in A} f(\omega(s+t)) \mid \mathcal{F}_t\Big)\Big) \\
&= \mathbf{1}_{\eta\in A} \mathbb{E}_\eta\Big(\mathbf{1}_{\omega(t),\omega(s+t)\in A} f(\omega(s+t))\Big),
\end{aligned}$$

where $\mathcal{F}_t$ is the σ-algebra containing all the rings until time t, use the Markov property and the tower property. Square brackets are used to emphasize that η is the argument to the result of the operators applied to f, and not the argument just to f. In particular, for $f = \mathbf{1}$ we find

$$e^{(t+s)\mathcal{L}^D}\mathbf{1}(\eta) = \mathbf{1}_{\eta\in A}\mathbb{P}_\eta(\omega(t),\omega(s+t)\in A).$$

This calculation immediately generalizes to arbitrarily many time steps so that, for a partition $S^{(i)} := [s_{i-1}, s_i)$ of $[0,t)$ with $i\in[0,k]$ for $k\in\mathbb{N}$ and $s_i \in [0,t)$ with $s_0 = 0$, $s_k = t$ and $s_{i-1} < s_i < s_{i+1}$ for $i \in [1, k-1]$, we find

$$e^{t\mathcal{L}^D}\mathbf{1}(\eta) = \prod_{i=1}^k e^{(s_i - s_{i-1})\mathcal{L}^D}\mathbf{1}(\eta) = \mathbf{1}_{\eta\in A}\mathbb{P}_\eta(\omega(s_i)\in A \forall i\in[1,k])\ .$$

In particular, since both k and the end-points s_i are arbitrary we find

$$e^{t\mathcal{L}^D}\mathbf{1}(\eta) = \mathbf{1}_{\eta\in A}\mathbb{P}_\eta(\tau > t).$$

The claim follows by taking the scalar product. □

Given a box Λ possibly depending on q, a simple consequence of Lemma 6.1 is that the hitting time τ_{x_Λ} satisfies

$$\mathbb{P}_\mu(\tau_{x_\Lambda} > t) \le e^{-\lambda^D(\Lambda)t},$$

where

$$\lambda^D(\Lambda) = \inf\{\mathcal{D}_\Lambda(f)/\mu_\Lambda(f^2) :\ f:\Omega_\Lambda \to \mathbb{R},\ f\restriction_{\{\omega:\,\omega_{x_\Lambda}=0\}} = 0\} \tag{6.1}$$

is the smallest eigenvalue for the Dirichlet problem

$$-\mathcal{L}_\Lambda f = \lambda f, \quad f\restriction_{\{\omega:\,\omega_{x_\Lambda}=0\}} = 0.$$

A lower bound on $\lambda^D(\Lambda)$ is obtained via the spectral gap $\gamma(\Lambda) > 0$ of the East process in Λ. For any f such that $f\restriction_{\{\omega:\omega_{x_\Lambda}=0\}} = 0$ we have

$$\begin{aligned}
\mathrm{Var}_\Lambda(f) &= \mathrm{Var}_\Lambda(f(1\cdot\omega_{\Lambda\setminus\{x_\Lambda\}})\mathbb{1}_{\omega_{x_\Lambda}=1}) \\
&= \mu_\Lambda(f^2(1\cdot\omega_{\Lambda\setminus\{x_\Lambda\}}))p - p^2\mu_\Lambda(f(1\cdot\omega_{\Lambda\setminus\{x_\Lambda\}}))^2
\end{aligned}$$

$$\begin{aligned}&\geq pq\mu_\Lambda(f^2(1\cdot\omega_{\Lambda\setminus\{x_\Lambda\}}))\\&= q\mu_\Lambda(f^2)\end{aligned}$$

where we used independence in the second equality and Jensen's inequality. Thus,

$$\lambda^D(\Lambda) \geq q\,\gamma(\Lambda). \tag{6.2}$$

Using Theorem 2.3 it follows that $\gamma(\Lambda) = \gamma_{d=1}^{(1+o(1))}$ as soon as $\max_i L_i \geq 2^{\theta_q}$ because of the slow relaxation mode along the edges of Λ on the coordinate axes.

If Λ is balanced, i.e. $\max_{i,j}(L_i \vee 1)/(L_j \vee 1) = O(1)$ as $q \to 0$, Equation (6.2) is a very pessimistic bound when $d \geq 2$ because $\lambda^D(\Lambda)$ should be mostly influenced by the *d-dimensional bulk dynamics* rather than by the *one-dimensional dynamics* along the edges of Λ. In this case it is natural to conjecture that, to the leading order as $q \to 0$, $\lambda^D(\Lambda)$ is lower bounded by γ_d. The following provides a better bound than Equation (6.2) in order to prove the conjecture.

Claim 6.2.

$$\begin{aligned}\lambda^D(\Lambda) &\geq \max\{\lambda^D(V) : V \subseteq \Lambda,\, V \supset \{0, x_\Lambda\}\}\\&\geq q\max\{\gamma(V) : V \subseteq \Lambda,\, V \supset \{0, x_\Lambda\}\} > 0.\end{aligned} \tag{6.3}$$

Proof of the claim. Clearly $\max\{\gamma(V) : V \subseteq \Lambda,\, V \supset \{0, x_\Lambda\}\} \geq \gamma(\Lambda) > 0$. Let $\Lambda \supseteq V \ni \{0, x_\Lambda\}$ and observe that monotonicity in the constraints implies that

$$\mathcal{D}_\Lambda(f) \geq \sum_{\omega\in\Omega_{\Lambda\setminus V}} \mu_{\Lambda\setminus V}(\omega)\mathcal{D}_V(f(\omega\,\cdot)),$$

where we recall that we leave away the explicit writing of the boundary condition to mean the boundary condition with no vacancy for both the Dirichlet form on Λ and V. Take now an $f : \Omega_\Lambda \to \mathbb{R}$ such that $f\restriction_{\{\omega:\,\omega_{x_\Lambda}=0\}} = 0$. For any subset $V \subset \Lambda$ such that $x_\Lambda \in V$, we have that $f(\omega\cdot\omega') = 0$ for any $\omega \in \Omega_{\Lambda\setminus V}$ and $\omega' \in \Omega_V$ if $\omega'_{x_\Lambda} = 0$. Therefore, Equation (6.1) implies for such f that

$$\mathcal{D}_V(f(\omega'\cdot)) \geq \lambda^D(V)\sum_{\omega''\in\Omega_V}\mu_V(\omega'')f^2(\omega'\cdot\omega'') = \lambda^D(V)\mu(f^2),$$

proving the first inequality of the claim. The second inequality follows from the general inequality Equation (6.2). $\square$

In order to bound from below the r.h.s. of Equation (6.3) according to whether $\max_{i,j}(L_i \vee 1)/(L_j \vee 1) = O(1)$ as $q \to 0$ or not, it is convenient to introduce the following geometrical definition.

Definition 6.3. Fix $d \geq 2, \beta \geq 0$, and $\kappa \geq 1$. For any given $q \in (0, 1)$ let $S(\beta, \kappa; \theta_q)$ be the collection of d-tuple of integers $(L_1, \ldots, L_d)$ such that $\max_{i,j}(L_i \vee 1)/(L_j \vee 1) \leq \kappa 2^{\beta\theta_q}$. We say that a box Λ with side lengths $(L_1, \ldots, L_d)$ is $(\beta, \kappa; \theta_q)$-squeezed if $(L_1, \ldots, L_d) \in S(\beta, \kappa; \theta_q)$.

In the sequel, the parameters β, κ will always be chosen independent of q. Moreover, whenever the value of q is understood we will simply write (β, κ)-squeezed instead of $(\beta, \kappa; \theta_q)$-squeezed.

Definition 6.4. Given $\beta \geq 0$ we say that $\lambda > 0$ satisfies condition $\mathcal{H}(\beta)$ and write $\lambda \sim \mathcal{H}(\beta)$ if for any $\kappa \geq 1, \varepsilon > 0$ there exists $q(\beta, \kappa, \varepsilon) > 0$ such that for each $q \leq q(\beta, \kappa, \varepsilon)$ the following occurs: For any $(\beta, \kappa; \theta_q)$-squeezed box Λ there exists a subset $V \subset \Lambda$ with $V \supset \{0, x_\Lambda\}$ such that $\gamma(V) \geq 2^{-(1+\varepsilon)\lambda\frac{\theta_q^2}{2}}$. We then let $\phi(\beta; d) = \min\{\lambda > 0 : \lambda \sim \mathcal{H}(\beta)\}$.

Thus, if $\lambda \sim \mathcal{H}(\beta)$ then Claim 6.2 implies that for all $\varepsilon > 0$ the Dirichlet eigenvalue $\lambda^D(\Lambda)$ is greater than $2^{-(1+\varepsilon)\lambda\frac{\theta_q^2}{2}}$ for *all* $(\beta, \kappa; \theta_q)$-squeezed box Λ and for all q small enough depending *only* on $\beta, \kappa, \varepsilon$. In particular,

$$\lambda^D(\Lambda) \geq 2^{-(1+\varepsilon)\phi(\beta;d)\frac{\theta_q^2}{2}}. \tag{6.4}$$

A major problem is then to bound the constant $\phi(\beta; d)$ for $d \geq 2$. Since the set of (β, κ)-squeezed boxes is an increasing set with increasing β, we immediately get monotonicity of $\phi(\beta; d)$.

Lemma 6.5. *Let $d \geq 2$ and $\beta > \beta' \geq 0$, then $\phi(\beta; d) \leq \phi(\beta'; d)$.*

So while it might be confusing for example that we call the equilateral cube (β, κ)-squeezed for any choice of $\beta \geq 0$ and $\kappa \geq 1$, to get bounds on $\phi(\beta; d)$ only the 'truly' (β, κ)-squeezed boxes matter, i.e. the ones that are (β, κ)-squeezed but not (β', κ')-squeezed for any $\beta' < \beta$ and any κ'.

Let us then come to the main concern of this chapter, which lies in bounding $\phi(\beta, d)$. Theorem 2.3 implies that $\phi(\beta, d) \leq 1$. The next result, is the technical core of this paper that lies behind the results of both problems treated in this thesis and the proof of which occupies the rest of this chapter.

Proposition 6.6. *For $d \geq 2$ the coefficient $\phi(\beta; d)$ satisfies:*

$$\begin{array}{ll} (i) & \phi(0; d) = 1/d, \\ (ii) & \phi(\beta; d) < 1 \quad \forall \beta \in (0, 1), \\ (iii) & \phi(\beta; d) = 1 \quad \forall \beta \geq 1. \end{array}$$

In particular, for any $d \geq 2$ and any (β, κ)-squeezed box Λ with $\beta < 1$ the Dirichlet eigenvalue $\lambda^D(\Lambda) \gg \gamma_{d=1}$ as $q \to 0$.

6.2 Dirichlet EV of balanced boxes: Proof of Proposition 6.6(i)

We proceed in two steps: we first prove that $\phi(0; d) \geq 1/d$ using a bottleneck argument and then that $\phi(0; d) \leq 1/d$.

The lower bound. Let Λ be the equilateral box of side length $L = \lfloor 2^{\theta_q/d} \rfloor$ and let $\Lambda \supset V \supset \{0, x_\Lambda\}$ be such that $\gamma(V) > 0$. We recall the construction of the bottleneck from [13, Section 4]. Given $\omega \in \Omega_\Lambda$ we define the gap $g_x(\omega)$ of $x \in \Lambda$ as

$$g_x(\omega) = (\|x\|_1 + 1) \wedge \min\{g > 0 \colon \exists z \in \Lambda \text{ with } z \prec x, \omega_z = 0, \|x - z\|_1 = g\}.$$

The minimum over the empty set is assumed to be ∞ here. We take the minimum with $\|x\|_1 + 1$ since the origin is always unconstrained, which is in contrast to the construction in [13] in which maximal boundary conditions were assumed. Starting from ω remove the vacancies from all vertices $x \in \Lambda$ with $g_x(\omega) = 1$, then remove all vacancies with gap two and so on until all vacancies with gap $dL - 1$ have been removed. Notice that removing all vacancies with gap dL or less always give rise to the configuration with no vacancies. Removing all vacancies with gap $dL - 1$ or less allows two different states, the one with no vacancies and the one with a single vacancy at x_Λ which we call $\mathbb{1}0$. Let $\mathcal{A}_* \subset \Omega_\Lambda$ be all configurations such that at the end of this procedure we are left with the state $\mathbb{1}0$.

Claim 6.7. $\forall\, \varepsilon > 0\; \exists\, q(\varepsilon) > 0$ *such that* $\forall\, q \leq q(\varepsilon)$ *we have* $\gamma(V) \leq 2^{-(1-\varepsilon)\frac{\theta_q^2}{2d}}$.

Proof. Let $A_V = \{\omega \in \Omega_V : \mathbb{1}_{V^c} \cdot \omega \in A_*\}$, where $\mathbb{1}_{V^c}$ denotes the configuration in Ω_{V^c} identically equal to one. We have $\mathbb{1}_V \notin A_V$ while the configuration with exactly one vacancy at x_Λ belongs to A_V. Therefore, $\mathrm{Var}\Big(\mathbb{1}_{A_V}\Big) \geq (1-q)^{2|V|-1}q = \Theta(q)$ because $|V| \leq 1/q$. Next we bound the Dirichlet form of $\mathbb{1}_{A_V}$. Let ∂A_V consists of those elements of A_V which are connected to A_V^c via a legal update for the East process on V. Then

$$\mathcal{D}_V(\mathbb{1}_{A_V}) \leq |\Lambda|\mu_V(\partial A_V) \leq |\Lambda|\mu_{V^c}(\mathbb{1}_{V^c})^{-1}\mu_\Lambda(\partial A_*) \leq 2^{-(1-o(1))\frac{\theta_q^2}{2d}},$$

where we used [13, Section 4.3] to bound $\mu_\Lambda(\partial A_*)$ which works in analogous fashion for minimal boundary conditions. The claim follows from the variational characterization of the spectral gap $\gamma(V)$. $\square$

Since the box Λ is $(0,1;\theta_q)$-squeezed, the claim implies that $\lambda \sim \mathcal{H}(0) \Rightarrow \lambda \geq 1/d$. Hence $\phi(0;d) \geq 1/d$.

The upper bound. The proof that $\phi(0;d) \leq 1/d$ requires a bootstrap procedure like the one introduced in [13]. The *base case* is Theorem 2.3 which gives that $\lambda = 1 \sim \mathcal{H}(0)$. We then prove the *recursive step*, namely that $\lambda \sim \mathcal{H}(0) \Rightarrow F(\lambda) \sim \mathcal{H}(0)$, where

$$F(\lambda) = ((2d-1)\lambda - 1)/(d^2\lambda - 1) < \lambda \quad \forall \lambda \in [1/d, 1]. \tag{6.5}$$

Since the mapping F has an attractive fixed point in $1/d$, the sought claim follows by iteration.

Proof of the recursive step The proof of the recursive step relies on the following coarse graining to lattices of boxes. In the sequel we assume that we have a fixed $q^* \in (0,1)$ and a family of finite probability spaces $\{\Omega_x^*, \mu_x^*\}_{x\in\mathbb{Z}_+^d}$ together with a family of *facilitating events* $\{G_x^* \subset \Omega_x^*\}_{x\in\mathbb{Z}_+^d}$ such that $\mu_x^*(G_x^*) = q^*$ for each $x \in \mathbb{Z}_+^d$.

Definition 6.8 (The *East chain). Let $V \subset \mathbb{Z}_+^d$ be a finite subset that contains the origin. The **East chain* on $\Omega_V^* := \otimes_{x\in V}\Omega_x^*$ with parameters $\{\Omega_x^*, \mu_x^*, q^*\}_{x\in V}$ is the continuous time Markov chain reversible w.r.t. $\mu_V^* = \otimes_{x\in V}\mu_x^*$ whose generator is given by $\mathcal{L}_V^* f(\omega) = \sum_{x\in V} c_x^*(\omega)[\mu_x^*(f) - f](\omega)$, where

$$c_x^*(\omega) = \begin{cases} 1 & \text{if } x = 0 \text{ or } \exists\, \mathbf{e} \in \mathcal{B} \text{ such that } x - e \in \Lambda \text{ and } \omega_{x-\mathbf{e}} \in G_{x-\mathbf{e}}^*, \\ 0 & \text{else.} \end{cases}$$

As in [13, Proposition 3.4] it is possible to prove that the spectral gap $\gamma^*(V)$ of the *East chain in V coincides with the spectral gap $\gamma(V;q^*)$ of the *standard* East chain with vacancy density q^*. We find it easier to work with *equilateral* boxes, i.e. $(0,1;\theta_q)$-squeezed boxes. For this purpose we first introduce a new condition, equivalent to $\mathcal{H}(0)$, which only requires a check on the spectral gap of suitable subsets of equilateral boxes.

Definition 6.9. We say that $\lambda \sim \mathcal{H}'(0)$ if $\forall\, \varepsilon > 0$ there exists $q(\varepsilon) > 0$ such that $\forall\, q \leq q(\varepsilon)$ and for any equilateral box Λ there exists $\Lambda \supset V \supset \{0, x_\Lambda\}$ such that $\gamma(V) \geq 2^{-\lambda(1+\varepsilon)\frac{\theta_q^2}{2}}$.

To prove the equivalence of $\mathcal{H}'(0)$ and $\mathcal{H}(0)$ we need a technical ingredient that we use frequently throughout this thesis so that we explicitly recall it here and give it a name. It is an upper bound on the variance if we don't have ergodic boundary conditions for the East model over a set V, but have a condition that ensures that if we extend the set V far enough at some point the East chain is going to be ergodic.

Lemma 6.10 (Enlargement trick, [13, Lemma 3.6]). *Let $\Lambda_x = \Lambda + x$ where Λ is a box of $\mathbb{Z}_+^d$ and x an arbitrary vertex. Let $V \subset \Lambda_x$ and let $A = \{\exists z \in \Lambda_x, z \prec V : \omega_z = 0\}$. Then,*

$$\mu_{\Lambda_x}\Big(\mathbb{1}_A \mathrm{Var}_V(f)\Big) \le \gamma(\Lambda)^{-1}\mathcal{D}_{\Lambda_x}(f).$$

With this we can prove the equivalence.

Lemma 6.11. $\lambda \sim \mathcal{H}'(0) \Leftrightarrow \lambda \sim \mathcal{H}(0)$.

Proof. Clearly, $\lambda \sim \mathcal{H}(0) \Rightarrow \lambda \sim \mathcal{H}'(0)$. Suppose that $\lambda \sim \mathcal{H}'(0)$, fix $\kappa \ge 1, \varepsilon > 0$ and let Λ be a $(0, \kappa, \theta_q)$-squeezed box with side lengths $(L_1, \ldots, L_d)$. Let $N = \min_j L_j$ and for any $i \in [d]$ choose a partition of the discrete interval $\{0, 1, \ldots, L_i\}$ into $N+1$ discrete intervals, $B_0^{(i)}, \ldots, B_N^{(i)}$, ordered from left to right, each one containing at least one vertex and at most $\kappa + 1$ vertices. For $\mathbf{j} \in \Lambda_B := \{0, \ldots, N\}^d$ write $B_{\mathbf{j}} = \prod_{i=1}^d B_{j_i}^{(i)}$ so that $\cup_{\mathbf{j} \in \Lambda_b} B_{\mathbf{j}} = \Lambda$. Furthermore, let $\Omega_{\mathbf{j}}^* := \Omega_{B_{\mathbf{j}}}$, $\mu_{\mathbf{j}}^* := \mu_{B_{\mathbf{j}}}$ and choose as facilitating event $G_{\mathbf{j}}$ the event that the smallest vertex in $B_{\mathbf{j}}$ in the $\prec$-ordering (for example the lowest-left corner if $d = 2$) has a vacancy. Clearly $\mu_{\mathbf{j}}^*(G_{\mathbf{j}}) = q \; \forall \mathbf{j} \in \Lambda_B$, i.e. $q^* = q$. Consider the *East chain on $\Omega_{\Lambda_B}^*$. Using $\lambda \sim \mathcal{H}'(0)$ there exists $V^* \subset \Lambda_B$ containing the origin and x_{Λ_B} such that

$$\gamma^*(V^*) = \gamma(V^*; q^*) = \gamma(V^*) \ge 2^{-\lambda(1+\varepsilon/2)\frac{\theta_q^2}{2}}.$$

Hence, if we set $V = \cup_{\mathbf{j} \in V^*} B_{\mathbf{j}}$ and write Var^* for the variance w.r.t. μ^* we get

$$\mathrm{Var}_V(f) = \mathrm{Var}_{V^*}^*(f) \le 2^{\lambda(1+\varepsilon/2)\frac{\theta_q^2}{2}} \sum_{\mathbf{j} \in V^*} \mu_V(c_{\mathbf{j}}^* \mathrm{Var}_{B_{\mathbf{j}}}(f)).$$

Using the enlargement trick (Lemma 6.10), Theorem 2.3 and the fact that each box $B_{\mathbf{j}}$ contains at most κ^d vertices, we get that the r.h.s. above is not larger than $2^{\lambda(1+\varepsilon/2)\frac{\theta_q^2}{2}}\, 2^{O(\kappa^d)\theta_q}\, \mathcal{D}_\Lambda(f)$ so that

$$\mathrm{Var}_V(f) \le 2^{\lambda(1+\varepsilon/2)\frac{\theta_q^2}{2}}\, 2^{O(\kappa^d)\theta_q} \mathcal{D}_V(f) \le 2^{\lambda(1+\varepsilon)\frac{\theta_q^2}{2}} \mathcal{D}_V(f).$$

Hence, for any q small enough depending on (ε, κ), $\gamma(V) \ge 2^{-\lambda(1+\varepsilon)\frac{\theta_q^2}{2}}$ implying that $\lambda \sim \mathcal{H}(0)$. □

Next, motivated by [13, Definition 5.2], we introduce another useful auxiliary Markov chain dubbed the **Knight Chain*. To that end, we first need to define the Knight graph (see Figure 6.1).

Definition 6.12 (The Knight graph). Given two vertices $x, y \in \mathbb{Z}^d$ we say that they form a Knight edge if there exists a $j \in [d]$ such that $y_i = x_i - 1$ for all $i \neq j$ and $y_j = x_j - 2$ or vice versa. We then consider the unique graph $G = (W, E), W \subset \mathbb{Z}^d$, constructed as follows. The vertex set W contains the origin and those $x \in \mathbb{Z}^d$ which are connected to the origin via a path of Knight edges. The edge set E consists of all the Knight edges of $W \times W$. It is easy to see that G is isomorphic to $\mathbb{Z}^d$ via the natural isomorphism Φ which is unique if we set $\Phi(0) = 0$.

The graph G will inherit the notation used so far for $\mathbb{Z}^d$ via the isomorphism Φ. We write $W_+ = \Phi^{-1}(\mathbb{Z}_+^d)$ and we say that $\Lambda^K \subset W_+$ is a Knight equilateral box containing the origin if $\Phi(\Lambda^K)$ is an equilateral box in $\mathbb{Z}_+^d$ containing the origin. In the latter case we write $x_{\Lambda^K} \in \Lambda^K$ for the vertex $\Phi^{-1}(x_{\Phi(\Lambda^K)})$. Notice that $\exists\, c > 0$ such that for any equilateral box $\Lambda \subset \mathbb{Z}_+^d$ containing the origin there exists a Knight equilateral box $\Lambda \supset \Lambda^K \ni 0$ such that $\|x_\Lambda - x_{\Lambda^K}\|_1 \le c$ (see Figure 6.1(B)).

Notice that that $\|z - z'\|_1 = d + 1$ for all $z, z' \in W$ connected by a Knight edge. For $x \in W$ let $E_x = \{y \in W^c : y \succ x,\ \|x - y\|_1 \le d\}$ be the enlargement of x. The enlargement of $V^K \subset W$ is the set $EV^K = \cup_{x \in V^K} E_x$.

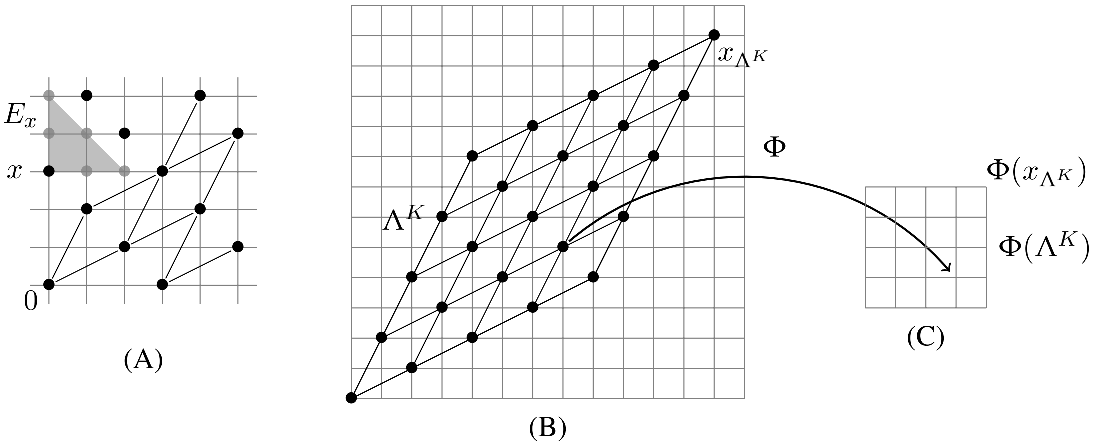

FIGURE 6.1 (A) A piece of the Knight graph (the black dots and the Knight edges) for $d = 2$. The gray triangle corresponds to the enlargement E_x of x. (B) The graph of the largest Knight equilateral box Λ^K of side length 4 inside an equilateral box of side length 13. (C) Under the natural isomorphism Φ the graph Λ^K becomes an equilateral box.

Definition 6.13 (The Knight and the *Knight chains)**.** Given an equilateral box Λ with origin at 0 and $V \subset \Lambda, V \ni 0$, let $V^K := \Phi^{-1}(V)$. Then the Knight chain on $\Omega^*_{V^K}$ is the image under Φ^{-1} of the *East chain on Ω_V. The *Knight chain on $\Omega_{EV^K\cap\Lambda}$ is the continuous time Markov chain evolving as follows. At any legal update at $z \in V^K$ of the Knight chain on $\Omega^*_{V^K}$ the whole configuration in $E_z \cap \Lambda$ is sampled from $\mu^*_{E_z\cap\Lambda}$.

Notice that we define this in general for the *East chain we fixed before Definition 6.8, the set of facilitating events that we use will be specified in the proof of Proposition 6.15. It is immediate to verify that the *Knight chain is reversible w.r.t. $\mu^*_{EV^K\cap\Lambda}$ with a positive spectral gap $\gamma^{*K}(EV^K \cap \Lambda)$, this spectral gap in turn is equal to the spectral gap of the standard East model on V with vacancy density q^* as we now verify.

Lemma 6.14. $\gamma^{*K}(EV^K \cap \Lambda) = \gamma(V; q^*)$.

Proof. Consider a partition $\{Q_x\}_{x\in V^K}$ of $(EV^K \setminus V^K) \cap \Lambda$ such that $Q_x \subset E_x$ for each x and such that the sets $\{Q_x\}_{x\in V^K}$ are mutually disjoint, a feature not necessarily shared by the sets $\{E_x \setminus \{x\}\}_{x\in V^K}$ (see Figures 6.1 and 6.2). Instead of the *Knight chain on $\Omega^*_{EV^K\cap\Lambda}$ consider the (very closely related) chain which at any legal update of the Knight chain at $x \in V^K$ resamples the whole configuration in $x \cup Q_x$. This chain can be viewed as a new Knight chain on $\Omega^*_{V^K}$ with new parameters $\tilde\Omega^*_x = \otimes_{z\in x\cup Q_x}\Omega^*_z, \tilde\mu^*_x = \otimes_{z\in x\cup Q_x}\mu^*_z, x \in V^K$, and the same facilitating events as the original Knight chain. Of course $\otimes_{x\in V^K}(\tilde\Omega^*_x, \tilde\mu^*_x) = (\Omega^*_{V^K}, \mu^*_{V^K})$. Hence, the spectral gap of the new chain, as discussed after Definition 6.8, coincides with $\gamma(V; q^*)$ and for all f

$$\begin{aligned}\mathrm{Var}^*_{V_K}(f) &\le \gamma(V;q^*)^{-1} \sum_{x\in V^K} \mu^*_{V^K}\Big(K_x \mathrm{Var}^*_{x\cup Q_x}(f)\Big)\\ &\le \gamma(V;q^*)^{-1} \sum_{x\in V^K} \mu^*_{V^K}\Big(K_x \mathrm{Var}^*_{E_x}(f)\Big).\end{aligned}$$

where $K_x = c_x \circ \Phi$ is the Knight constraint at x. Above we used the fact that K_x does not depend on $\{\omega_z\}_{z\in x\cup Q_x}$ and that in average $\mu^*_{E_x}\Big(\mathrm{Var}^*_{x\cup Q_x}(f)\Big) \le \mathrm{Var}^*_{E_x}(f)$ (see Lemma 9.15 for a slightly more general form of this). The sum in the r.h.s. above is the Dirichlet form of the *Knight chain and we conclude that its spectral gap is at least $\gamma(V; q^*)$. The reverse inequality follows immediately by projection onto the variables $\eta_x = 1 - \mathbb{1}_{\{\omega_x \in G_x^*\}}, x \in V^K$, where G_x^* is the facilitating event. $\square$

We can finally state the main result of this section.

Proposition 6.15. *Fix $\lambda \in (1/d, 1]$ and let $F(\cdot)$ be the mapping in Equation* (6.5). *Then $\lambda \sim \mathcal{H}'(0) \Rightarrow F(\lambda) \sim \mathcal{H}'(0)$.*

Proof. Let $\lambda \in (1/d, 1]$ with $\lambda \sim \mathcal{H}'(0)$ and let $\Lambda \subset \mathbb{Z}_+^d$ be an equilateral box with side length L. Using a suitable λ-dependent *Knight chain, we will now construct a set $V \subset \Lambda$ such that $\gamma(V) \geq 2^{-F(\lambda)\frac{\theta_{q*}^2}{2}(1+\varepsilon)}$.

Let $\ell = \lfloor 2^{m\theta_q} \rfloor$, where $m = (d\lambda - 1)/(d^2\lambda - 1)$ and observe that $\ell \leq 2^{\theta_q/d}$. If $L \leq \ell$ we can use Theorem 2.3 to get that

$$\gamma(\Lambda) \geq 2^{-(m-m^2/2)\theta_q^2(1+o(1))} \geq 2^{-F(\lambda)\frac{\theta_q^2}{2}(1+o(1))}.$$

In this case we simply choose $V = \Lambda$. If instead $L > \ell$ we proceed as follows. Let B_0 be the

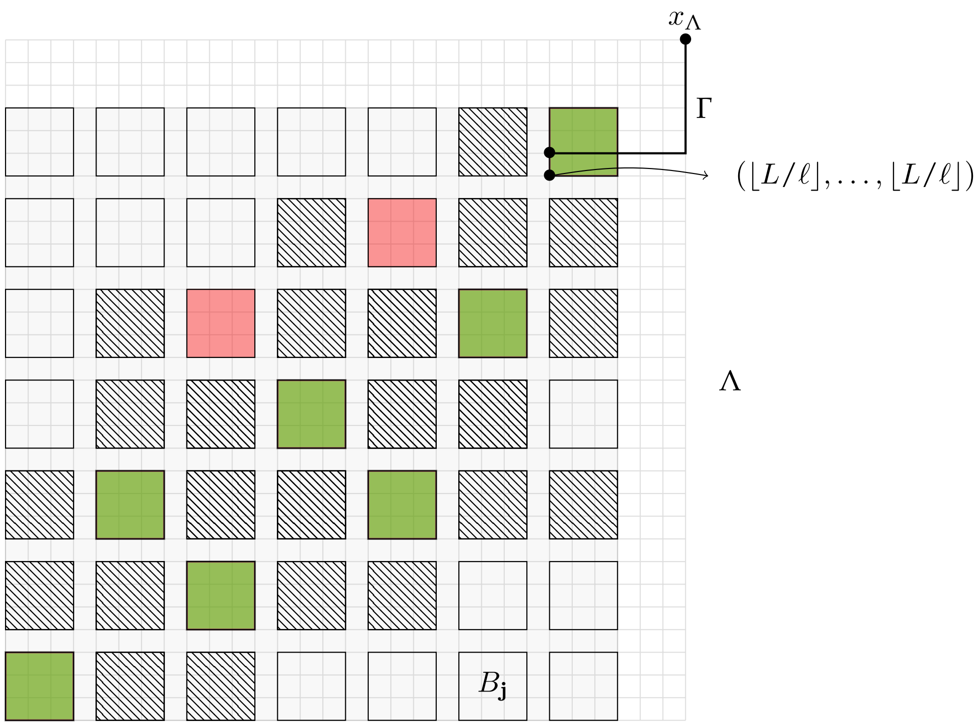

FIGURE 6.2 The setting in the proof of Proposition 6.15 with $\ell = 3$ and $L = 30$. The 3×3 boxes $B_{\mathbf{j}}$ are those with $\mathbf{j} \in \Lambda_B$, the coloured (red/green) ones are those with $\mathbf{j} \in \Lambda_B^K$, the green ones are those with $\mathbf{j} \in V^K$, and the dashed ones are those with $\mathbf{j} \in (EV^K \cap \Lambda_B) \setminus V^K$. The set V with $\gamma(V) \geq 2^{-F(\lambda)\frac{\theta_q^2}{2}(1+o(1))}$ is the union of the green and dashed boxes together with the path Γ.

equilateral box with side length ℓ, let $\Lambda_B := \{0, \ldots, \lfloor L/\ell \rfloor\}^d$ and for $\mathbf{j} \in \mathbb{Z}_+^d$ let $B_{\mathbf{j}} = B_0 + \mathbf{j}\ell$. Thus $\cup_{\mathbf{j} \in \Lambda_B} B_{\mathbf{j}} \subset \Lambda$ and $\min_{x \in B_{\mathbf{j}_{\Lambda_B}}} \|x - x_\Lambda\|_1 \leq O(\ell)$. We say that $B_{\mathbf{j}}$ is *good* if it contains at least one vacancy and observe that the density $q^* = 1 - (1-q)^{\ell^d}$ of good boxes satisfies

$$q\ell^d/2 \le q^* \le q\ell^d \le 1 \quad \Rightarrow \quad \theta_{q^*} \in [\theta_q(1-dm), \theta_q(1-dm)+1],$$

where we use the Bonferroni inequality for the lower bound. In the sequel we will use the Knight chain and the *Knight chain with $\Omega^*_{\mathbf{j}} = \{0,1\}^{B_{\mathbf{j}}}$, $\mu^*_{\mathbf{j}} = \otimes_{x \in B_{\mathbf{j}}} \mu_x$, and facilitating events $G^*_{\mathbf{j}} = \{B_{\mathbf{j}} \text{ is good}\}$.

Let $\Lambda_B^K \subset \Lambda_B$ be the largest Knight equilateral box containing the origin and for $V^K \subset \Lambda_B^K$ consider the *Knight chain on $\Omega_{EV^K \cap \Lambda_B}$. Using $\lambda \sim \mathcal{H}'(0)$ we can choose $V^K \subset \Lambda_B^K$ such that $V^K \supset \{0, \mathbf{j}_{\Lambda_B^K}\}$ and such that $\forall \varepsilon > 0$ and q small enough depending on ε we have

$$\gamma^{*K}(EV^K \cap \Lambda_B) = \gamma(\Phi(V^K); q^*) \ge 2^{-\lambda \frac{\theta_{q^*}^2}{2}(1+\epsilon/2)}, \tag{6.6}$$

where in the equality we used Lemma 6.14. We then take $V = V_1 \cup \Gamma \subset \Lambda$, where $V_1 = \cup_{\mathbf{j} \in EV^K \cap \Lambda_B} B_{\mathbf{j}}$ and $\Gamma = (x^{(0)}, x^{(1)}, \dots, x^{(N)})$ is any path in Λ satisfying: (i) $x^{(0)} \in V_1$, $x^{(N)} = x_\Lambda$, (ii) $x^{(i-1)} \prec x^{(i)}$ $\forall i \in [N]$, and (iii) $N = O(\ell)$. By construction such a path always exists.

Claim 6.16. *For any $\varepsilon > 0$ there exists $q(\varepsilon) > 0$ such that for all $q \le q(\varepsilon)$*

$$\gamma(V) \ge 2^{-\frac{1}{2}(\lambda \theta_{q^*}^2 + (2m - m^2)\theta_q^2)(1+\epsilon)} = 2^{-F(\lambda)\frac{\theta_q^2}{2}(1+\epsilon)}.$$

Clearly the claim proves the proposition. □

The proof of the claim requires another of our often used tools, so let us again explicitly recall it and give it a name. Consider two sets $V_1, V_2 \subset \mathbb{Z}^d$ together with some state space $\Omega^{(i)}$ and measures $\nu^{(i)}$ on V_i for $i \in [2]$. Write $V = V_1 \cup V_2$ and $\nu = \nu^{(1)} \otimes \nu^{(2)}$. Consider an event $\mathcal{A}$ on $\Omega^{(1)}$ such that $\nu^{(1)}(\mathcal{A}) > 0$. The first result gives a Poincaré-inequality for the block KCM on V where V_1 is unconstrained and every ring the state is sampled according to $\nu^{(1)}$. On V_2 the rings are legal if V_1 is in state $\mathcal{A}$ and if this is the case then the state on V_2 is sampled according to $\nu^{(2)}$.

Lemma 6.17 (Block relaxation Lemma, [11, Proposition 4.4])**.** *In the above situation we have for $f : \Omega^{(1)} \otimes \Omega^{(2)} \to \mathbb{R}$*

$$\mathrm{Var}_\nu(f) \le \frac{2}{\nu^{(1)}(\mathcal{A})} \nu(\mathrm{Var}_{\nu^{(1)}}(f) + \mathbb{1}_{\mathcal{A}} \mathrm{Var}_{\nu^{(2)}}(f)).$$

We can now proceed with the proof of Claim 6.16.

Proof of the claim. Fix $\varepsilon > 0$ and choose q small enough depending on ε. Let $V_2 = \Gamma \setminus \{x^{(0)}\}$. Using the block relaxation Lemma gives

$$\mathrm{Var}_V(f) \le 2^{\theta_q + 1} \mu(\mathrm{Var}_{V_1}(f) + \mathbb{1}_{\omega_{x^{(0)}} = 0} \mathrm{Var}_{V_2}(f)). \tag{6.7}$$

Given the boundary constraint $\sigma \in \Omega_{\partial_\downarrow^+ V_2}$ that consists of a unique vacancy at $x^{(0)}$ the one-dimensional East model on V_2 is ergodic so using Theorem 2.3 we have

$$\begin{aligned}\mu(\mathbb{1}_{\omega_{x^{(0)}}=0} \mathrm{Var}_{V_2}(f)) &\le 2^{(2m-m^2)\frac{\theta_q^2}{2}(1+\varepsilon)} \sum_{i=1}^{N} \mu\Big[\mathbb{1}_{\omega_{x^{(i-1)}}=0} \mathrm{Var}_x(f)\Big] \\ &\le 2^{(2m-m^2)\frac{\theta_q^2}{2}(1+\varepsilon)} \mu(\mathcal{D}_{V_2}(f)),\end{aligned}$$

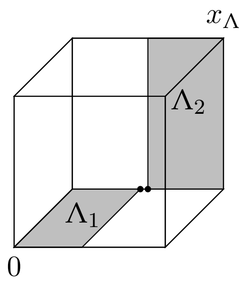

FIGURE 6.3 The boxes Λ_1, Λ_2. The two black dots denote x_{Λ_1} and the origin of Λ_2 at $x_{\Lambda_1} + \mathbf{e}_1$ respectively.

where we use that $\mathbb{1}_{\omega_{x(i-1)}=0} \le c_x^V$, and thus $\gamma^\sigma(V_2) \ge 2^{-(2m-m^2)\frac{\theta_q^2}{2}(1+\varepsilon)}$. Moreover, using the enlargement trick (Lemma 6.10) together with Equation (6.6) and the fact that $\gamma(B_0) \ge 2^{-(2m-m^2)\frac{\theta_q^2}{2}(1+\varepsilon)}$ gives $\gamma(V_1) \ge 2^{-\frac{1}{2}(\lambda\theta_{q*}^2+(2m-m^2)\theta_q^2)(1+\epsilon)}$. Finally, observe that $\gamma(V) \ge 2^{-\theta_q-2}\min(\gamma(V_1), \gamma^\sigma(V_2))$ and

$$(\lambda\theta_{q_*}^2 + (2m-m^2)\theta_q^2) = F(\lambda)\theta_q^2(1+o(1)) \quad \text{as } q \to 0$$

so that the claim follows. □

Remark 6.18. Note that in the lower bound of $\gamma(V)$ instead of $2^{-\theta_q-1}$ we have $2^{-\theta_q-2}$ which we did not further elaborate on as it is absorbed into the ε in the claim anyway. The term appears because if $V_1 \cap V_2 \neq \emptyset$ then any vertex $x \in V_1 \cap V_2$ appears once in both upper bounds of the variances in Equation (6.7). We call this the *overcounting of* x and while it is a relatively benign contribution here, controlling it is of vital importance for the lower bounds of the spectral gap of the MCEM process in Chapter 9.

The recursive step $\lambda \sim \mathcal{H}(0) \Rightarrow F(\lambda) \sim \mathcal{H}(0)$ now follows immediately from Proposition 6.15 and Lemma 6.11.

6.3 Dirichlet EV of slightly unbalanced boxes: Proof of Proposition 6.6(ii)

The proof consists of two steps. We first prove that $\phi(\beta; 2) < 1$ for all $\beta < 1$ implies that the same holds for any $d \ge 3$ by induction and then we deal with the initial two-dimensional case.

6.3.1 The induction step

Fix $d \ge 3$ and $\beta < 1$ and assume $\phi(\beta; d') < 1$ for any $2 \le d' \le d-1$. We are going to prove that $\phi(\beta; d) < 1$ as well. Fix $\kappa \ge 1$ together with a (β, κ)-squeezed box Λ with side lengths $(L_1, \dots, L_d)$ and set (see Figure 6.3)

$$\Lambda_1 = \{x \in \Lambda : \ x_1 \le \lfloor L_1/2 \rfloor, \ x_d = 0\},$$
$$\Lambda_2 = \{x \in \Lambda : \ x_1 > \lfloor L_1/2 \rfloor, x_i = L_i, \ 2 \le i \le d-1\}.$$

By construction, the origin of the box Λ_2 is at $x_{\Lambda_1} + \mathbf{e}_1$ and $x_{\Lambda_2} = x_\Lambda$. Moreover, both Λ_1 and Λ_2 are (β, κ)-squeezed boxes in $\mathbb{Z}_+^{d-1}$ and $\mathbb{Z}_+^2$ respectively. The induction hypothesis implies that for all $\varepsilon > 0$ and all q small enough depending on $\varepsilon, \beta, \kappa$ there exist $V_i \subset \Lambda_i, i = 1, 2$, such that

- $V_1 \supset \{0, x_{\Lambda_1}\}$ and $V_2 \supset \{x_{\Lambda_1} + \mathbf{e}_1, x_\Lambda\}$;
- $\gamma(V_1) \geq 2^{-(1+\varepsilon)\phi(\beta;d-1)\frac{\theta_q^2}{2}}$ and $\gamma^\sigma(V_2) \geq 2^{-(1+\varepsilon)\phi(\beta;2)\frac{\theta_q^2}{2}}$, where $\sigma \in \Omega_{\partial_\downarrow^+ V_2}$ has a unique vacancy at x_{Λ_1}.

Using the block relaxation Lemma analogously to how it was used in the proof of Claim 6.16 then implies that $\gamma(V) \geq 2^{-(1+2\varepsilon)(\phi(\beta;d-1)\vee\phi(\beta;2))\frac{\theta_q^2}{2}}$, i.e. $\phi(\beta;d) \leq \phi(\beta;d-1) \vee \phi(\beta;2)) < 1$.

6.3.2 The base case $d = 2$

We will prove that for any $\beta \in (0,1)$

$$\phi(\beta;2) \leq \frac{1}{2}(1-\beta)^2 + 2\beta - \beta^2, \tag{6.8}$$

which, in particular, implies that $\phi(\beta;2) < 1$ for any $\beta < 1$. The main idea here is to partition a (β,κ)-squeezed box Λ into suitably chosen mesoscopic boxes in such a way that the coarse-grained version of Λ becomes a $(0,2)$-squeezed box on which the control of the Dirichlet eigenvalue gap is assured by part (i) of the proposition.

Fix $0 < \beta < 1, \kappa \geq 1$ together with a (β,κ)-squeezed box Λ with side lengths (L_1, L_2), and assume w.l.o.g. that $L_1 = \min_i L_i$. We set $\ell = \lceil (L_2+1)/2(L_1+1) \rceil \leq (\kappa/2)2^{\beta\theta_q}$, and by Lemma 6.5 we can even assume $\ell = \Theta(2^{\beta\theta_q})$. Assume further w.l.o.g. that $(L_2+1)/\ell \in \mathbb{N}$. We then partition Λ into vertical one dimensional boxes $B_\mathbf{j} = B + x_\mathbf{j}$, $B = \{0\} \times \{0, \ldots, \ell-1\}, x_\mathbf{j} = (j_1, j_2\ell)$ where $\mathbf{j} \in Q = \{0, \ldots, L_1 - 1\} \times \{0, \ldots, (L_2+1)/\ell - 1\}$. We also write $\Omega_\mathbf{j}^*, \mu_\mathbf{j}^*$ for $\Omega_{B_\mathbf{j}}$ and $\mu_{B_\mathbf{j}}$ respectively.

Let $\tilde{Q}$ be the subset of Q lying between the two 45°-lines, one through the origin and the other through the point x_Q and declare that $\mathbf{j}, \mathbf{j}' \in \tilde{Q}$ form an edge if either $j_2 = j_2' + 1$ and $j_1 \in \{j_1', j_1' + 1\}$ or vice versa (see Figure 6.4). The corresponding graph over the vertex set $\tilde{Q}$ is isomorphic via the natural graph isomorphism Φ to the box $\Phi(\tilde{Q}) \subset \mathbb{Z}_+^2$ with origin at $x = 0$ and side lengths $L_1 - 1, (L_2+1)/\ell - L_1$. In particular, we write $\mathbf{j}' \prec \mathbf{j}$ iff $\Phi(\mathbf{j}') \prec \Phi(\mathbf{j})$.

On any subset V of $\tilde{Q}$ we consider the image of the *East chain on $\Phi(V)$ (or rather a slightly altered version of it as we see below) with parameters $\Omega_\mathbf{j}^*, \mu_\mathbf{j}^*$ and facilitating event $G_\mathbf{j} = \{\omega_{B_\mathbf{j}} \neq 1\}$. Thus $q^* = 1 - (1-q)^\ell$ and $\theta_{q^*} = (1-\beta)\theta_q + \Theta(1)$. As the box $\Phi(\tilde{Q})$ is $(0,2)$-squeezed, part (i) of Proposition 6.6 implies the existence of $W \subset \Phi(\tilde{Q})$, containing the origin and $x_{\Phi(\tilde{Q})}$ such that, for any $\varepsilon > 0$ and any q sufficiently small depending on ε,

$$\gamma(W;q^*) \geq 2^{-(1+\varepsilon/2)\frac{\theta_{q^*}^2}{4}}. \tag{6.9}$$

Recall the definition of enlargements E_x from above Definition 6.13. We define $E\Phi^{-1}(W) := \cup_{\mathbf{j}\in\Phi^{-1}(W)} E_\mathbf{j} \cap Q$ and $V = \cup_{\mathbf{j}\in E\Phi^{-1}(W)} B_\mathbf{j} \subset \Lambda$ and observe that V contains the origin and the vertex x_Λ.

Claim 6.19. *For any $\varepsilon > 0$ and any q sufficiently small depending on ε*

$$\gamma(V) \geq \gamma(W;q^*) \times 2^{-(\beta-\beta^2/2)\theta_q^2(1+\varepsilon)}.$$

The claim together with Equation (6.9) finally implies that $\gamma(V) \geq 2^{-((1-\beta)^2/2+2\beta-\beta^2)\frac{\theta_q^2}{2}(1+\varepsilon)}$, $\forall\, q \leq q(\varepsilon)$, i.e. Equation (6.8). □

The proof of the claim is analogous to the proof of Lemma 6.14. For the Knight chain the enlargements were necessary to build a connected V (i.e. to interpolate between the green boxes in Figure 6.2). Here,

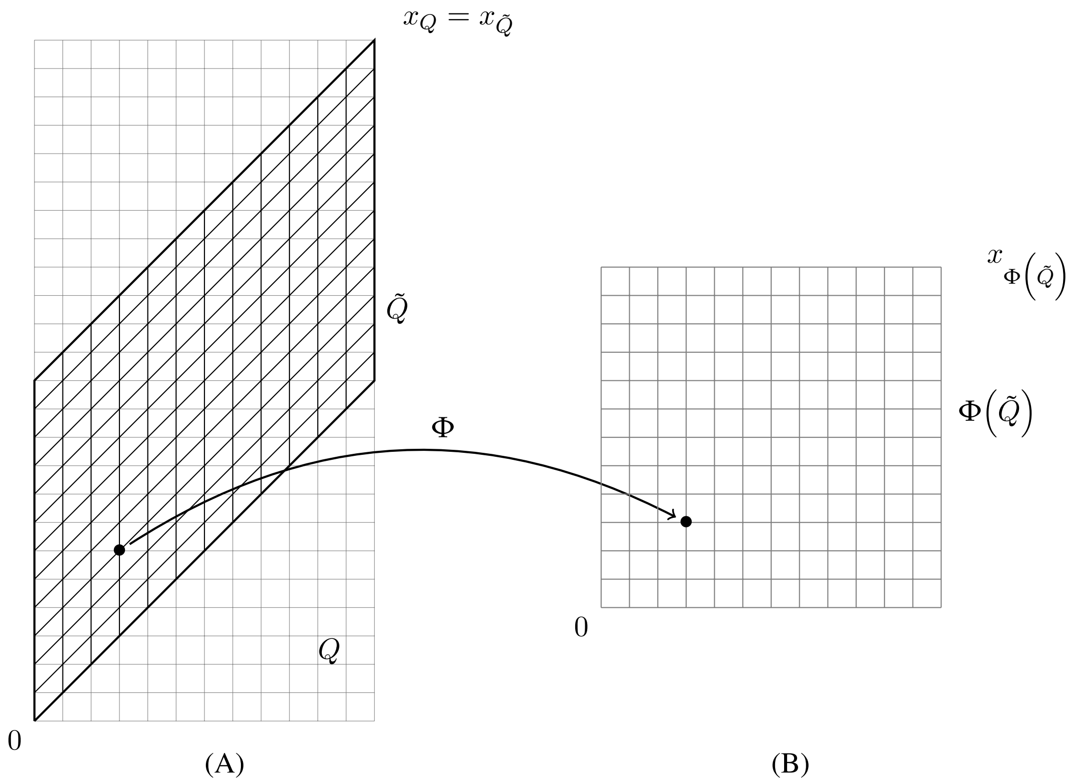

FIGURE 6.4 (A). The box Q and the region $\tilde{Q}$ with its graph structure. Each vertex $\mathbf{j} \in Q$ represents the box $B_{\mathbf{j}}$. (B). Under the natural isomorphism Φ the graph $\tilde{Q}$ becomes an equilateral square in $\mathbb{Z}^2$.

$\Phi^{-1}(W)$ is already connected, but using enlargements allows us to use the enlargement trick to get Equation (6.11) below, rather than proving an analogous result to the enlargement trick applying to the unenlarged $\Phi^{-1}(W)$.

Proof of the claim. On V we define an auxiliary dynamics to the *East chain. Consider for that a partition of $E\Phi^{-1}(W)$ into disjoint connected subsets $U_{\mathbf{j}}$ for $\mathbf{j} \in \Phi^{-1}(W)$ such that $\mathbf{j} \in U_{\mathbf{j}} \subset E_{\mathbf{j}}$ and $\cup_{\mathbf{j}\in\Phi^{-1}(W)} U_{\mathbf{j}} = E\Phi^{-1}(W)$. In the sequel we write $B_{U_{\mathbf{j}}} := \cup_{\mathbf{j}'\in U_{\mathbf{j}}} B_{\mathbf{j}'}$ and analogously for $B_{E_{\mathbf{j}}}$. For the constraints with $c^*_{\mathbf{j}}(\omega) = 1$ iff either $\mathbf{j} = 0$ or there exists a neighbor $\mathbf{j}' \prec \mathbf{j}$ such that there exists at least a vacancy in $B_{\mathbf{j}'}$ (note: not $B_{U_{\mathbf{j}'}}$). For such constraints we define the auxiliary dynamics that update $B_{U_{\mathbf{j}}}$ with a configuration sampled from $\mu_{B_{U_{\mathbf{j}}}}$ if $c^*_{\mathbf{j}}(\omega) = 1$ and otherwise do nothing. The spectral gap of this chain is, as the one for the enlarged East chain, given by $\gamma(W, q^*)$, since the $\mathbf{j}$ that participate in the dynamics are only the ones in $\Phi^{-1}(W)$. The Poincaré inequality reads

$$\mathrm{Var}_V(f) \leq \gamma(W; q^*)^{-1} \sum_{\mathbf{j}\in\Phi^{-1}(W)} \mu_V\left(c^*_{\mathbf{j}} \mathrm{Var}_{B_{U_{\mathbf{j}}}}(f)\right), \quad \forall f. \tag{6.10}$$

We now bound a generic term $\mu_V\left(c^*_{\mathbf{j}}(\omega)\mathrm{Var}_{B_{U_{\mathbf{j}}}}(f)\right)$. Using the enlargement trick (Lemma 6.10), Theorem 2.3, and $\ell \leq O(\kappa)2^{\beta\theta_q}$, for any $\varepsilon > 0$ and any q small enough depending on ε we get

$$\mu_V\left(c^*_{\mathbf{j}}(\omega)\mathrm{Var}_{B_{U_{\mathbf{j}}}}(f)\right) \leq 2^{(\beta-\beta^2/2)\theta_q^2(1+\varepsilon/2)} \sum_{\substack{z\in B_{E_{\mathbf{j}'}} \\ \mathbf{j}'=\mathbf{j} \text{ or } \mathbf{j}'\prec\mathbf{j}, \|\mathbf{j}'-\mathbf{j}\|=1}} \mu_V\left(c^V_z \mathrm{Var}_z(f)\right). \tag{6.11}$$

By combining Equation (6.10) and Equation (6.11) and using that $|E_{\mathbf{j}}| = O(\ell)$ (implying that the overcounting is of the same order), we conclude for q small enough that

$$\mathrm{Var}_V(f) \le \gamma(W; q^*)^{-1} \times 2^{(2\beta-\beta^2)\frac{\theta_q^2}{2}(1+\varepsilon)} \mathcal{D}_V(f) \quad \forall f,$$

and the claim follows from the variational characterization of $\gamma(V)$. □

6.4 Dirichlet EV of unbalanced boxes: Proof of Proposition 6.6(iii)

We already know (cf. Theorem 2.3) that $\phi(\beta; d) \le 1 \; \forall \beta$. Fix $\beta \ge 1$ and consider the $(\beta, 1)$-squeezed one dimensional box $\Lambda = \cup_{k=0}^{\lfloor 2^{\beta\theta_q} \rfloor} \{k\, \mathbf{e}_1\}$. The only subset $V \subset \Lambda$ containing the origin and x_Λ and such that $\gamma(V) > 0$ is $V = \Lambda$. But $\gamma(\Lambda) = 2^{-\frac{\theta_q^2}{2}(1+o(1))}$ (see again Theorem 2.3) so that $\phi(\beta; d) \ge 1$.

6.5 Spectral gap maximizing subsets of small boxes

Recall that we say that $\lambda \sim H(\beta)$ if we have the corresponding lower bound on the Dirichlet Eigenvalue for *all* $(\beta, \kappa; \theta_q)$-squeezed Λ. The way we proved this is by showing that there is a subset V of any such Λ such that the spectral gap of the East chain with minimal boundary conditions on this V satisfies this lower bound. If instead of any Λ first we look only at a single $(0, \kappa; \theta_q)$-squeezed Λ (or rather a family $\{\Lambda_q\}_{q \in (0,1)}$ to be exact) we get the following more nuanced result.

Proposition 6.20. *Let Λ be a $(0, \kappa; \theta_q)$-squeezed box of side lengths $(L_1, \ldots, L_d)$ with $\min_i L_i \in (2^{n-1}, 2^n]$ and $n = n(q)$ with $\lim_{q\to 0} n(q) = \infty$. We have that as $q \to 0$ we can find a subset $V \subset \Lambda$ with $\{0, x_\Lambda\} \subset V$ such that*

$$\gamma(V, q) = \begin{cases} 2^{-(n\theta_q - n^2 + O(n\theta_q^{1/2}))} & : n < \theta_q/d, \\ 2^{-(\theta_q^2/2d + O(\theta_q^{3/2}/d))} & : \textit{else}. \end{cases}$$

if q small enough.

Sketch of proof for $d = 2$. The proof is analogous to Proposition 6.6(i) and, in fact, implies it and only the proof for the lower bound changes (the upper bound still follows from the bottleneck).

Recall that the original version of Theorem 2.3, i.e. [13, Theorem 2] contained more precise lower order terms in the spectral gap with minimal boundary conditions, than we have cited here. Using the same construction as in the proof of Lemma 6.11 we can w.l.o.g. assume that Λ is equilateral. Starting from a $V \subset \Lambda$ such that

$$\gamma(V, q) \ge \begin{cases} 2^{-(n\theta_q - n^2 + \lambda n^2 + n\log_2(n) + O(\theta_q^{1/2}))} & : n < \theta_q/2, \\ 2^{-(\lambda\theta_q^2 + \theta_q \log_2(\theta_q) + O(\theta_q))} & : \text{else}, \end{cases}$$

we want to identify a subset $V' \subset V$ such that the same holds with $\lambda \mapsto \tilde{F}(\lambda)$ where

$$\tilde{F}(\lambda) = \begin{cases} \lambda/(1+2\lambda) & : n < \theta_q/2, \\ F(\lambda) & : \text{else}. \end{cases}$$

The proof for $n > \theta_q/2$ is unchanged (see Proposition 6.15). For $n < \theta_q/2$ we get that with each iteration $\lambda \to \tilde{F}(\lambda)$ the coefficient decreases and we can repeat the proof completely analogously starting from the $n < \theta_q/2$ bound and using $m = (1 + 2\lambda n)/(1 + 2\lambda)$. □

Chapter 7

Front evolution: proofs

In this chapter we prove Theorems 1 and 3. In Section 7.1 we present two tools that enter the proofs of these theorems. The first tool is an upper bound on the hitting time of the top-right corner of boxes of given side length. Using Proposition 6.6 we show that this hitting time is governed by the two-dimensional mode, along the diagonal, of the East process. The second result is an adaptation of the bottleneck from [13, Section 4] that we already used in Chapter 6 and which gives us a lower bound on a different, but related, hitting time.

These tools allow us to say that the two-dimensional modes are dominating in the proof of Theorem 1(A), by giving the corresponding upper bound on $v_{\max}$ and lower bound on $v_{\min}$. The proof for Theorem 1(B) is completely analogous only that in this case the two-dimensional mode is not dominant anymore (i.e. we cannot use Proposition 6.6(i)) but still dominant enough for the front speed to be distinguishably different from the front speed of the one-dimensional mode (i.e. we use Proposition 6.6(ii)). For Theorem 1(C) we construct a set which allows us to quantify how much the two-dimensional and how much the one-dimensional mode enters the propagation speed giving the speed depending on α given in the statement.

In Section 7.3 we use similar techniques to establish the region behind the front in which the process is mixing and close the section in Section 7.4 with the proof of the cutoff.

7.1 Two key tools

Let us thus come to the two auxiliary results described above.

7.1.1 Upper bounds on the hitting times

The next result is a direct consequence of Proposition 6.6 and Lemma 6.1.

Lemma 7.1. *Fix $\beta \geq 0, \kappa \geq 1$, and let $\Lambda = \Lambda_q$ be a $(\beta, \kappa; \theta_q)$-squeezed box of side lengths $(L_1, \dots, L_d)$ such that $2^{\theta_q^{3/2}}/2 \leq \min_i L_i \leq 2^{\theta_q^{3/2}}$. Then, uniformly in $x \in \mathbb{Z}_+^d$, the following holds. For any $\varepsilon > 0 \; \exists \, q(\varepsilon, \beta, \kappa)$ such that for any $q \leq q(\varepsilon, \beta, \kappa)$*

$$\sup_{\omega \in \{\omega\colon \omega_x = 0\}} \mathbb{E}_\omega(\tau_{x+x_\Lambda}) \leq 2^{(1+\varepsilon)\phi(\beta;d)\frac{\theta_q^2}{2}}.$$

Proof. Fix $x \in \mathbb{Z}_+^d, \varepsilon > 0$ and let $T(\varepsilon) = 2^{(1+\varepsilon)\phi(\beta;d)\frac{\theta_q^2}{2}}$, $T^* = 2^{2\theta_q^2}$. Then

$$\mathbb{E}_\omega(\tau_{x+x_\Lambda}) = \int_0^{T(\varepsilon)} dt \, \mathbb{P}_\omega(\tau_{x+x_\Lambda} > t) + \int_{T(\varepsilon)}^{T^*} dt \, \mathbb{P}_\omega(\tau_{x+x_\Lambda} > t)$$

$$+\int_{T^*}^{+\infty} dt\, \mathbb{P}_\omega(\tau_{x+x_\Lambda} > t)$$

$$\leq T(\varepsilon) + T^*\mathbb{P}_\omega(\tau_{x+x_\Lambda} > T(\varepsilon)) + \int_{T^*}^{+\infty} dt\, \mathbb{P}_\omega(\tau_{x+x_\Lambda} > t). \tag{7.1}$$

We will now prove that the supremum over $\omega \in \{\omega : \omega_x = 0\}$ of the second and third term in the r.h.s. of Equation (7.1) tend to zero as $q \to 0$.

Given $\ell \in \mathbb{N}$ let $V_{x,\ell} = \{x_1 - \ell, \dots, x_1\} \times \cdots \times \{x_d - \ell, \dots, x_d\} \cap \mathbb{Z}_+^d$ and let $\mathcal{G}(t,\ell), t > 0$, be the event that there exists $z \in V_{x,\ell}$ such that

$$\mathcal{T}_t(z) = \int_0^t ds\, 1_{\{c_z(\omega(s))=1\}} > t/\ell^d. \tag{7.2}$$

In other words z is unconstrained for a fraction ℓ^{-d} of the time t. When such a vertex exists we will write $\xi \in V_{x,\ell}$ for the smallest one in the lexicographical order[1]. In [15, Corollary 4.2] it has been proved[2] that there exist constants $c, c' > 0$ such that

$$\sup_{\omega \in \{\omega :\, \omega_x = 0\}} \mathbb{P}_\omega(\mathcal{G}(t,\ell)^c) \leq c' t \ell^d e^{-cq\ell}. \tag{7.3}$$

Armed with the above result we now deal with the term $T^*\mathbb{P}_\omega(\tau_{x+x_\Lambda} > T(\varepsilon))$ in the r.h.s. of Equation (7.1). Fix $\omega \in \{\omega : \omega_x = 0\}$ and choose $\ell = \lfloor \frac{1}{2} \min_i L_i \rfloor$. All the bounds proven below will be uniform in ω. Using Equation (7.3)

$$T^*\mathbb{P}_\omega(\tau_{x+x_\Lambda} > T(\varepsilon)) \leq c'(T^*)^2 \ell^d\, e^{-cq\ell} + T^*\, \mathbb{P}_\omega(\tau_{x+x_\Lambda} > T(\varepsilon);\, \mathcal{G}(T(\varepsilon),\ell)). \tag{7.4}$$

The assumption $\min_i L_i = \Theta\Big(2^{\theta_q^{3/2}}\Big)$ and the choice of ℓ imply that the first term in the r.h.s. above is $o(1)$ as $q \to 0$.

Next, write $\mathbb{P}_\omega(\tau_{x+x_\Lambda} > T(\varepsilon);\, \mathcal{G}(x,\ell)) = \sum_{y \in V_{x,\ell}} \mathbb{P}_\omega(\tau_{x+x_\Lambda} > T(\varepsilon); \xi = y)$ and let $\Lambda_y = \{y_1, \dots, x_1 + L_1\} \times \cdots \times \{y_d, \dots, x_d + L_d\}$. Notice that the choice of ℓ together with the fact that Λ is (β,κ)-squeezed imply that Λ_y is $(\beta, \kappa+1)$-squeezed. Let $\mathcal{F}_y$ be the σ-algebra generated by the variables $\{\omega_z(s) : z \in \partial_\downarrow^+ \Lambda_y, s \leq T(\varepsilon)\}$ and observe that $\{c_y(\omega(s))\}_{s \leq T(\varepsilon)}$ is measurable w.r.t. $\mathcal{F}_y$. Then

$$\mathbb{P}_\omega(\tau_{x+x_\Lambda} > T(\varepsilon); \xi = y) = \mathbb{E}_\omega(1_{\{\xi = y\}} \mathbb{P}_\omega(\tau_{x+x_\Lambda} > T(\varepsilon) \,|\, \mathcal{F}_y)).$$

The orientation of the East process implies that, conditionally on $\mathcal{F}_y$, the event $\{\tau_{x+x_\Lambda} > T(\varepsilon)\}$ coincides with the same event for the *time-inhomogeneous* East chain in Ω_{Λ_y} with *deterministic, time-dependent* boundary conditions on $\partial_\downarrow^+ \Lambda_y$. We denote the law of the latter chain with initial state $\omega \restriction_{\Lambda_y}$ by $\hat{\mathbb{P}}_\omega(\cdot)$. Thus,

$$\begin{aligned}
\mathbb{P}_\omega(\tau_{x+x_\Lambda} > T(\varepsilon) \,|\, \mathcal{F}_y) &= \hat{\mathbb{P}}_\omega(\tau_{x+x_\Lambda} > T(\varepsilon)) \\
&\leq \mu(\omega \restriction_{\Lambda_y})^{-1} \sum_{\eta \in \Omega_{\Lambda_y}} \mu(\eta) \hat{\mathbb{P}}_\eta(\tau_{x+x_\Lambda} > T(\varepsilon)) \\
&\leq 2^{\theta_q |\Lambda_y|} \sum_{\eta \in \Omega_{\Lambda_y}} \mu(\eta) \hat{\mathbb{P}}_\eta(\tau_{x+x_\Lambda} > T(\varepsilon)).
\end{aligned} \tag{7.5}$$

[1] The lexicographical order here means a total order version of the $\prec$-partial order where we say that x is larger than y if for the smallest $i \in [d]$ such that $x_i \neq y_i$ we have $x_i \geq y_i$.

[2] Technically the proof concerns the East model on $\mathbb{Z}^d$ not on $\mathbb{Z}_+^d$ with an unconstrained origin, but the proof carries over.

Let $t_0 \equiv 0 < t_1 < t_2 < \cdots < t_n < t_{n+1} \equiv T(\varepsilon)$ be the times at which the boundary conditions on $\partial_\downarrow^+ \Lambda_y$ change and let $\sigma^{(i)}, i \in [n+1]$, denote the boundary condition during the time interval (t_{i-1}, t_i). Let also $\hat{\mathcal{L}}^{(i)}$ be the generator of the East chain on Ω_{Λ_y} with boundary conditions $\sigma^{(i)}$ and let $\mathcal{A}^{(i)} = 1_{A^c} \hat{\mathcal{L}}^{(i)} 1_{A^c}$ be the generator $\hat{\mathcal{L}}^{(i)}$ with Dirichlet boundary condition on $A = \{\eta \in \Omega_{\Lambda_y} :\ \eta_{x+x_\Lambda} = 0\}$.

Then apply Lemma 6.1 with Λ_y, A^c, and $\hat{\mathcal{L}}$ the generator for the time-inhomogeneous chain. Split the exponential into the smaller time steps on which the chain is homogeneous, thus recovering $\hat{\mathcal{L}}_i$. This can be done since $e^{t\hat{\mathcal{L}}}$ is a probability semi-group and for $t \in [t_{i-1}, t_i]$, $e^{t\hat{\mathcal{L}}}$ is the probability semi-group for the time-homogeneous chain with boundary conditions $\sigma^{(i)}$. Thus,

$$\sum_{\eta \in \Omega_{\Lambda_y}} \mu_{\Lambda_y}(\eta) \hat{\mathbb{P}}_\eta(\tau_{x+x_\Lambda} > T(\varepsilon)) = \langle \mathbf{1}_A, e^{t_1 \mathcal{A}^{(1)}} \times e^{(t_2 - t_1)\mathcal{A}^{(2)}} \times \cdots \times e^{(t_{n+1} - t_n)\mathcal{A}^{(n+1)}} \mathbf{1}_A \rangle.$$

Let $\lambda_i \geq 0$ be the smallest eigenvalue of $-\mathcal{A}^{(i)}$. Clearly,

$$\langle \mathbf{1}, e^{t_1 \mathcal{A}^{(1)}} \times e^{(t_2 - t_1)\mathcal{A}^{(2)}} \times \cdots \times e^{(t_{n+1} - t_n)\mathcal{A}^{(n+1)}} \mathbf{1} \rangle \leq e^{-\sum_{i=1}^{n+1} (t_i - t_{i-1})\lambda_i}.$$

If during the time interval (t_i, t_{i+1}) the constraint c_y at the vertex y is zero then we simply use $\lambda_i \geq 0$. If instead $c_y = 1$ we use monotonicity of λ_i in the boundary conditions $\sigma^{(i)}$ together with Equation (6.4), the definition of $\phi(\beta; d)$, and the fact that Λ_y is $(\beta, \kappa + 1)$-squeezed to get that

$$\lambda_i \geq \lambda^D(\Lambda_y) \geq 2^{-(1+\varepsilon/2)\phi(\beta;d)\frac{\theta_q^2}{2}},$$

for all q small enough depending only on $\varepsilon, \beta, \kappa$. In conclusion, recalling that

$$\int_0^{T(\varepsilon)} ds\, 1_{\{c_y = 1\}} \geq T(\varepsilon)\ell^{-d},$$

we get

$$\begin{aligned}
&\langle \mathbf{1}, e^{t_1 \mathcal{A}^{(1)}} \times e^{(t_2 - t_1)\mathcal{A}^{(2)}} \times \cdots \times e^{(t_{n+1} - t_n)\mathcal{A}^{(n+1)}} \mathbf{1} \rangle \\
&\qquad \leq \exp\Big(- \Big(\int_0^{T(\varepsilon)} ds\, 1_{\{c_y = 1\}} \Big) \times\ 2^{-(1+\varepsilon/2)\phi(\beta;d)\frac{\theta_q^2}{2}} \Big) \\
&\qquad \leq \exp\Big(- T(\varepsilon)\ell^{-d}\, 2^{-(1+\varepsilon/2)\phi(\beta;d)\frac{\theta_q^2}{2}} \Big) \\
&\qquad \leq \exp\Big(- 2^{\varepsilon\phi(\beta;d)\frac{\theta_q^2}{4}} \Big),
\end{aligned}$$

for $q \leq q(\varepsilon, \beta, \kappa)$. Putting all together we obtain that the second term in the r.h.s. of Equation (7.4) is bounded from above by

$$2^{2\theta_q^2} \times 2^{\theta_q 2^{O(\theta_q^{3/2})}} \times e^{-2^{\varepsilon\phi(\beta;d)\frac{\theta_q^2}{4}}} = o(1) \quad \text{as } q \to 0.$$

We finally examine the third term in the r.h.s. of Equation (7.1). In order to bound from above $\mathbb{P}_\omega(\tau_{x+x_\Lambda} > t), t \geq T^*$, we proceed exactly as above except that now the parameter ℓ of the box $V_{x,\ell} \subset \mathbb{Z}_+^d$ has to be chosen depending on t, $\ell_t = t^{1/4d}$. Using the same notation we get

$$\mathbb{P}_\omega(\tau_{x+x_\Lambda} > t) \leq c' t \ell_t^d e^{-cq\ell_t} + 2^{\theta_q(\ell_t + \max_i L_i)^d - t\ell_t^{-d} \min_{y \in V_{x,\ell_t}} \lambda^D(\Lambda_y)}. \tag{7.6}$$

Notice that if t is so large that V_{x,ℓ_t} coincides with the box of side lengths $(x_1,\dots,x_d)$, then the event $\mathcal{G}^c(t,\ell_t)=\emptyset$ because the origin is always unconstrained. In this case the first term in the r.h.s. above is absent while the second term becomes $2^{\theta_q \max_i (x_i+L_i)^d-\lambda^D(\Lambda_x)t}$ where Λ_x is the box with side lengths $(x_1+L_1,\dots,x_d+L_d)$.

Back on track, in order to bound from below $\min_{y\in V_{x,\ell_t}}\lambda^D(\Lambda_y)$ we are no longer allowed to appeal to Proposition 6.6 because the box Λ_y could be extremely squeezed in some directions and we are forced the use the spectral gap bound, Equation (3.3),

$$\min_y \lambda^D(\Lambda_y)\ge 2^{-(1+\varepsilon)\frac{\theta_q^2}{2}}.$$

In conclusion, we obtain

$$\begin{aligned}\mathbb{P}_\omega(\tau_{x+x_\Lambda}>t)&\le c' t\ell_t^d e^{-cqt^{1/4d}}+e^{O(\theta_q)t^{1/4}-t^{3/4}2^{-\frac{\theta_q^2}{2}(1+\varepsilon)}}\\&\le c' t\ell_t^d e^{-cqt^{1/4d}}+e^{-t^{3/4}2^{-\frac{\theta_q^2}{2}(1+\varepsilon)}/2}\quad \forall\, t>T^*.\end{aligned}$$

We finally observe that

$$\int_{T^*}^{+\infty} dt\,\left[c' t\ell_t^d e^{-cqt^{1/4d}}+e^{-t^{3/4}2^{-\frac{\theta_q^2}{2}(1+\varepsilon)}/2}\right]=o(1)\quad \text{as } q\to 0.$$

□

7.1.2 The bottleneck on scale $2^{\frac{\theta_q}{d}}$ with maximal boundary conditions

Recall the bottleneck from [13, Section 4] that we used in the lower bound in the proof of Proposition 6.6(i) in Section 6.2. We can use the analogous bottleneck with maximal boundary condition, as it was defined originally in [13, Section 4], as opposed to the minimal boundary conditions used in Section 6.2 to get a bound on the hitting time τ_{x_Λ} started from a state with no vacancy in the equilateral box Λ with side length $L\le 2^{\theta_q/d}$.

Definition 7.2 (Legal path)**.** A sequence $(\omega^{(1)},\dots,\omega^{(n)})$ of configurations (in Ω or $\Omega_\Lambda,\Lambda\subset\mathbb{Z}^d_+$,) such that $\omega^{(i+1)}$ is obtained from $\omega^{(i)}$ by means of a (non-trivial) legal update will be referred to as a *legal path* joining $\omega^{(1)}$ to $\omega^{(n)}$.

Before discussing the core of this section, we point out the following *monotonicity property of legal updates*. Take two sets $\Lambda\subset\Lambda'\subset\mathbb{Z}^d_+$ together with two boundary conditions σ,σ' on $\partial^+_\downarrow\Lambda$ and $\partial^+_\downarrow\Lambda'$ respectively such that $\sigma_x=0\;\forall\, x\in\partial^+_\downarrow\Lambda\cap\Lambda'$ and $\sigma_x\le\sigma'_x\;\forall\, x\in\partial^+_\downarrow\Lambda\cap\partial^+_\downarrow\Lambda'$. Then a legal update at $x\in\Lambda$ of the East process in Λ' with boundary condition σ' is a legal update at x of the East process in Λ with boundary condition σ.

Definition 7.3 (Bottleneck)**.** Let $\Lambda_L=\{0,\dots,L\}^d$, and for $x\in\mathbb{Z}^d_+\setminus\Lambda_L$ let $V_{x,L}=(\Lambda_L+x-x_{\Lambda_L})\cap\mathbb{Z}^d_+$. We say that $A\subset\Omega_{V_{x,L}}$ is an (x,L)-*bottleneck* if any legal path joining $E_{x,L}\equiv\{\omega\in\Omega:\omega\restriction_{V_{x,L}}=1\}$ with $\{\omega:\omega_x=0\}$ hits $\{\omega:\omega\restriction_{V_{x,L}}\in A\}$.

Proposition 7.4. *In the setting of Definition 7.3 $\forall\varepsilon>0\;\exists\, q(\varepsilon)>0$ such that for $q\le q(\varepsilon)$ the following holds. For any $L\le 2^{\theta_q/d}$ and $x\in\mathbb{Z}^d_+\setminus\Lambda_L$ there exists a (x,L)-bottleneck A with $\mu(A)\le 2^{-(n\theta_q-d\binom{n}{2})(1-\varepsilon)}$ where $n:=\lfloor\log_2(L)\rfloor$.*

Proof. Fix $L \leq 2^{\theta_q/d}$ and $x \in \mathbb{Z}_+^d \setminus \Lambda_L$, and w.l.o.g. suppose that $V_{x,L} \subset \mathbb{Z}_+^d$. The case when this assumption fails follows immediately from the monotonicity property of legal updates described above. Fix also $\omega \in E_{x,L}$, $\hat{\omega} \in \{\omega : \omega_x = 0\}$, together with a legal path $\Gamma = (\omega^{(1)}, \ldots, \omega^{(k)})$ joining them. Finally, write $\omega_V^{(j)} = \omega^{(j)} \restriction_{V_{x,L}}, j \in [k]$, and let $1 \leq j_1 < j_2 < \cdots < j_m \leq k$ be those indices such that the legal update occurs inside $V_{x,L}$. Using the monotonicity of legal updates the sequence $\hat{\Gamma} = (\omega_V^{(j_1)}, \ldots, \omega_V^{(j_m)})$ connects $\omega_{V_{x,L}} \equiv 1$ to $\{\omega \in \Omega_{V_{x,L}} : \omega_x = 0\}$ via legal updates of the East chain in $V_{x,L}$ with *maximal* boundary conditions. The results of [13, Section 4] then imply that $\hat{\Gamma}$ must hit a fixed subset A of $\Omega_{V_{x,L}}$ whose equilibrium probability satisfies the required bound. □

Corollary 7.5. *In the same setting*

$$\max_{\omega \in E_{x,L}} \mathbb{P}_\omega(\tau_x < t) \leq O(t) \times 2^{-(n\theta_q - d\binom{n}{2})(1-\varepsilon)}.$$

Notice that for $L = 2^{\theta_q/d}$ *the r.h.s. above becomes equal to* $O(t) \times 2^{-\frac{\theta_q^2}{2d}(1-\epsilon)}$.

Proof. We only give a quick sketch because the proof of similar statements has already appeared elsewhere (see e.g. [14]). Fix $L \leq 2^{\theta_q/d}$ and $x \in \mathbb{Z}_+^d \setminus \Lambda_L$. Using Proposition 7.4 there exists $A \subset \Omega_{V_{x,L}}$ such that

$$\max_{\omega \in E_{x,L}} \mathbb{P}_\omega(\tau_x < t) \leq \max_{\omega \in E_{x,L}} \mathbb{P}_\omega(\tau_A \leq t).$$

For a given $\omega \in \Omega_{V_{x,L}^c}$ write $\delta_\omega \otimes \mu_{V_{x,L}}$ for the product measure on Ω whose marginals on $\Omega_{V_{x,L}^c}$ and $\Omega_{V_{x,L}}$ are the Dirac mass at ω and $\mu_{V_{x,L}}$ respectively. Using $L \leq 2^{\theta_q/d}$ we get $\mu_{V_{x,L}}(\omega \restriction_{V_{x,L}} = 1)^{-1} = O(1)$ as $q \to 0$. Hence we can do a large deviation bound on the number of rings in $V_{x,L}$ and a union bound in the rings to get,

$$\begin{aligned}\max_{\omega \in E_{x,L}} \mathbb{P}_\omega(\tau_A \leq t) &\leq O(1) \times \max_{\omega \in \Omega_{V_{x,L}^c}} \mathbb{P}_{\delta_\omega \otimes \mu_{V_{x,L}}}(\tau_A \leq t) \\ &\leq O(t\, L^d) \max_{\omega \in \Omega_{V_{x,L}^c}} \sup_{s \leq t} \mathbb{P}_{\delta_\omega \otimes \mu_{V_{x,L}}}(\omega(s) \restriction_{V_{x,L}} \in A).\end{aligned}$$

It is easy to check (see [15, Section 3]) that $\mu_{V_{x,L}}$ is stationary for the marginal on $\Omega_{V_{x,L}}$ of the East process with initial distribution $\delta_\omega \otimes \mu_{V_{x,L}}$. Hence, the r.h.s. above is equal to

$$O(tL^d)\mu(A) \leq O(t) 2^{-(n\theta_q - d\binom{n}{2})(1-2\varepsilon)}$$

for q small enough depending on ε. □

7.2 Front velocity bounds: Proof of Theorem 1

We split the proofs for the various parts into their own subsections.

7.2.1 Bulk velocity: Proof of Theorem 1(A)

In the sequel $\mathbf{x} \in \mathbb{R}_+^d$ will denote a unit vector independent of q with $\min_i \mathbf{x}_i > 0$.

Lower bound on $v_{\min}(\mathbf{x})$

Let $\ell = \lfloor 2^{\theta_q^{3/2}} \rfloor$ and let $x^{(n)} = \lfloor n\ell\mathbf{x} \rfloor$, $n \in \mathbb{N}$. We begin by proving that

$$\limsup_{n\to\infty} \frac{\mathbb{E}_{\omega^*}(\tau_{x^{(n)}})}{n} \le 2^{\frac{\theta_q^2}{2d}(1+o(1))} \quad \text{as } q \to 0. \tag{7.7}$$

Clearly

$$\tau_{x^{(n+1)}} \le \inf\{s \ge \tau_{x^{(n)}} : \omega_{x^{(n+1)}}(s) = 0\},$$

so that, using the strong Markov property,

$$\mathbb{E}_{\omega^*}(\tau_{x^{(n+1)}}) \le \mathbb{E}_{\omega^*}(\tau_{x^{(n)}}) + \max_{\omega \in \{\omega :\, \omega_{x^{(n)}} = 0\}} \mathbb{E}_{\omega}(\tau_{x^{(n+1)}}).$$

Let $L_i = x_i^{(n+1)} - (x_i^{(n)} + 1)$, $i \in [d]$. Clearly the box with sides length $(L_1, \dots, L_d)$ is $(0, \kappa)$-squeezed with $\kappa = \max_{i,j} \mathbf{x}_i / \mathbf{x}_j + 1$ and Lemma 7.1 implies that, uniformly in n, for any $\varepsilon > 0$

$$\max_{\omega \in \{\omega :\, \omega_{x^{(n)}} = 0\}} \mathbb{E}_{\omega}(\tau_{x^{(n+1)}}) \le 2^{\frac{\theta_q^2}{2d}(1+\varepsilon)}, \tag{7.8}$$

for any q sufficiently small depending on ε. Equation (7.7) now follows immediately.

In order to complete the proof of (A) we write

$$\mathbb{E}_{\omega^*}(\tau_{n\mathbf{x}}) \le \mathbb{E}_{\omega^*}(\tau_{x^{(\lfloor n/\ell \rfloor)}}) + \max_{\omega \in \{\omega :\, \omega_{x^{(\lfloor n/\ell \rfloor)}} = 0\}} \mathbb{E}_{\omega}(\tau_{n\mathbf{x}}).$$

By using the arguments entering into the proof of Lemma 7.1 it is easy to see that

$$\sup_n \max_{\omega \in \{\omega :\, \omega_{x^{(\lfloor n/\ell \rfloor)}} = 0\}} \mathbb{E}_{\omega}(\tau_{n\mathbf{x}}) < +\infty.$$

Therefore,

$$\limsup_{n\to\infty} \frac{\mathbb{E}_{\omega^*}(\tau_{n\mathbf{x}})}{n} \le \ell^{-1} 2^{\frac{\theta_q^2}{2d}(1+o(1))} = 2^{\frac{\theta_q^2}{2d}(1+o(1))},$$

because of the choice of ℓ. In conclusion we have proved that $v_{\min}(\mathbf{x}) \ge 2^{-\frac{\theta_q^2}{2d}(1+o(1))}$ as $q \to 0$. $\square$

Upper bound on $v_{\max}(\mathbf{x})$.

Let $\ell = \lfloor 2^{\theta_q/d} \rfloor$ and for any $y \in \mathbb{Z}_+^d$ with $\|y\|_1 \ge \ell$ let $\Lambda_y = \{z \in \mathbb{Z}_+^d : z \prec y,\ \|y - z\|_1 \le \ell\}$. Clearly, if the starting configuration of the East process on $\mathbb{Z}_+^d$ is ω^*, then $\tau_{\partial_\downarrow^+ \Lambda_y} < \tau_y$ a.s. Hence, for all $\lambda > 0$ the strong Markov property gives

$$\begin{aligned}\mathbb{E}_{\omega^*}(e^{-\lambda \tau_y}) &= \mathbb{E}_{\omega^*}\Big(e^{-\lambda \tau_{\partial_\downarrow^+ \Lambda_y}} \mathbb{E}_{\omega_{\tau_{\partial_\downarrow^+ \Lambda_y}}}(e^{-\lambda \tau_y})\Big) \\ &\le W(\lambda) \sum_{z \in \partial_\downarrow^+ \Lambda_y} \mathbb{E}_{\omega^*}(e^{-\lambda \tau_z}),\end{aligned} \tag{7.9}$$

where $W(\lambda) := \sup_{z :\, \|z\|_1 \ge \ell} \max_{\omega \in \{\omega :\, \omega\restriction_{\Lambda_z} = 1\}} \mathbb{E}_{\omega}(e^{-\lambda \tau_z})$. Iterate Equation (7.9) using $|\partial_\downarrow^+ \Lambda_y| \le O(\ell^{d-1})$ to get that

$$\mathbb{E}_{\omega^*}(e^{-\lambda \tau_y}) \le \Big(O(\ell^{d-1}) W(\lambda)\Big)^{\lfloor \|y\|_1 / \ell \rfloor}.$$

Claim 7.6. *For any $\varepsilon > 0$ sufficiently small let $T(\varepsilon) = 2^{\frac{\theta_q^2}{2d}(1-\varepsilon)}$ and choose $\lambda = \lambda(\varepsilon, q) = \varepsilon\theta_q^2 T(\varepsilon)^{-1}$. Then $W(\lambda(\varepsilon, q)) \leq e^{-\Omega(\varepsilon\theta_q^2)}$ as $q \to 0$.*

Proof of the claim. Using Corollary 7.5, for any z with $\|z\|_1 \geq \ell$ and any q small enough depending on ε, we get

$$\begin{aligned}\max_{\omega\in\{\omega:\,\omega\restriction_{\Lambda_z}=1\}} \mathbb{E}_\omega(e^{-\lambda\tau_z}) &\leq e^{-\lambda T(\varepsilon)} + \max_{\omega\in\{\omega:\,\omega\restriction_{\Lambda_z}=1\}} \mathbb{P}_\omega(\tau_z \leq T(\varepsilon)) \\ &\leq e^{-\varepsilon\theta_q^2} + O(T(\varepsilon))2^{-\frac{\theta_q^2}{2d}(1-\varepsilon/2)} = e^{-\Omega(\varepsilon\theta_q^2)}.\end{aligned}$$

□

Using Jensen's inequality, $e^{-\lambda\mathbb{E}_{\omega^*}(\tau_y)} \leq \mathbb{E}_{\omega^*}(e^{-\lambda\tau_y})$, and choosing λ as in the claim, we finally obtain

$$\mathbb{E}_{\omega^*}(\tau_y) \geq \Omega\Big(2^{\frac{\theta_q^2}{2d}(1-\varepsilon)}\Big)\lfloor 2^{-\theta_q/d}\|y\|_1\rfloor. \tag{7.10}$$

In particular, Equation (7.10) implies that $v_{\max}(\mathbf{x}) \leq 2^{-\frac{\theta_q^2}{2d}(1-o(1))}$ as $q \to 0$. □

Remark 7.7. Exactly the same proof applies to get the following result. For any $\varepsilon > 0$ there exists $q(\varepsilon) > 0$ and $c(\varepsilon) > 0$ such that the following holds for $q \leq q(\varepsilon)$. For any $y \in \mathbb{Z}_+^d$ and $n \leq \|y\|_1$ let $B(y, n) = \{z :\ z \prec y,\ \|y - z\|_1 \leq n\}$. Then

$$\max_{\omega:\omega\restriction_{B(y,n)}=1} \mathbb{P}_\omega(\tau_y \leq nT(\varepsilon)) \leq e^{-c\varepsilon\theta_q^2\lfloor n2^{-\frac{\theta_q}{d}}\rfloor}.$$

7.2.2 Approaching the axis slowly: Proof of Theorem 1(B)

The proof is identical to that of Section 7.2.1 with the following modification. The box Λ with side lengths $L_i = x_i^{(n+1)} - (x_i^{(n)} + 1), i \in [d]$, is now $(\beta, \kappa + 1)$-squeezed because of the assumption on the direction $x = x(q)$. Using again Lemma 7.1 we get the analogue of Equation (7.8):

$$\max_{\omega\in\{\omega:\,\omega_{x^{(n)}}=0\}} \mathbb{E}_\omega(\tau_{x^{(n+1)}}) \leq 2^{\phi(\beta;d)\frac{\theta_q^2}{2}(1+\varepsilon)}.$$

The rest of the argument remains unchanged and the conclusion is that

$$\limsup_{n\to\infty} \frac{\mathbb{E}_{\omega^*}(\tau_{nx})}{n} \leq \ell^{-1}2^{\phi(\beta;d)\frac{\theta_q^2}{2}(1+\varepsilon)},$$

i.e.

$$\limsup_{q\to 0} -\frac{1}{\theta_q^2}\log_2(v_{\min}(x)) \leq \frac{\phi(\beta; d)}{2} < \frac{1}{2} \tag{7.11}$$

because $\phi(\beta; d) < 1$ if $\beta \in [0, 1)$. □

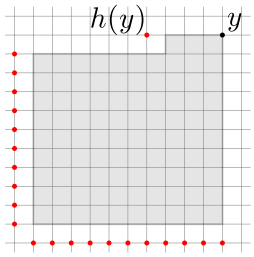

FIGURE 7.1 Example for a set U_y (the grey region). The red vertices denote $\partial_{\downarrow}^{+} U_y$.

7.2.3 Approaching the axis quickly: Proof of Theorem 1(C)

Fix a q-dependent unit vector $\mathbf{x} \in \mathbb{R}_+^2$ such that $0 < \mathbf{x}_2 \le \mathbf{x}_1 2^{-\theta_q^2 \alpha}$ with $\alpha > 0$. In order to track how a vacancy can propagate from the origin to the vertex $\lfloor n\mathbf{x} \rfloor \in \mathbb{Z}_+^2$ we introduce the following construction.

Let $0 < \varepsilon \ll 1$ and let $L = L(\varepsilon, \alpha, q) = \lfloor 2^{\theta_q^2 \alpha (1-\varepsilon/2)} \rfloor$. W.l.o.g. we assume that q is so small that $L \gg 2^{\theta_q}$.

Definition 7.8. For $y = (y_1, y_2) \in \mathbb{Z}_+^2$ such that $1 \le y_2 \le 2^{-\theta_q^2 \alpha} y_1$ let $B_{y,L} \subset \mathbb{Z}^2$ be the box of side lengths (L, L) and upper-right corner at y and let (see Figure 7.1)

$$U_y = (B_{y,L} \setminus \cup_{i=\lfloor 1/q \rfloor + 1}^{L} \{y - i\mathbf{e}_1\}) \cap \mathbb{Z}_+^2.$$

Let also $h(y) := y - (\lfloor 1/q \rfloor + 1)\mathbf{e}_1$ and note that $h(y) \in \partial_{\downarrow}^{+} U_y$.

If the starting configuration of the East process on $\mathbb{Z}_+^2$ is ω^*, then $\tau_{\partial_{\downarrow}^{+} U_y} < \tau_{U_y} < \tau_y$. This observation justifies the following definition. In the sequel $\{\omega_t\}_{t \ge 0}$ denotes the East process in $\mathbb{Z}_+^2$ with $\omega_0 = \omega^*$.

Definition 7.9 (Infection sequence for y)**.** Let $\xi^{(0)} = y$ and define recursively $\xi^{(i)}$ as the unique vertex $z \in \partial_{\downarrow}^{+} U_{\xi^{(i-1)}}$ such that $\omega_{\tau_{\partial_{\downarrow}^{+} U_{\xi^{(i-1)}}}}(z) = 0$. We also let $\nu := \inf\{i \in \mathbb{N} \colon 0 \in U_{\xi^{(i)}}\}$ and call the random sequence $\xi(y) = \{\xi^{(i)}\}_{i \in [\nu]}$ the *infection sequence for* y. The collection of all possible infection sequences is denoted by $\mathcal{S}(y)$. Given $\mathbf{v} = \{v^{(i)}\}_i \in \mathcal{S}(y)$ we say that $v^{(i)}$ is *good* if $v^{(i+1)} = h(v^{(i)})$ and *bad* otherwise.

Remark 7.10. By construction any possible infection sequence $\mathbf{v}$ is such that $\|v^{(i)} - v^{(i+1)}\|_1 \ge \lfloor 1/q \rfloor$.

Lemma 7.11. *For any q small enough, any infection sequence in $\mathcal{S}(y)$ contains at most y_2 bad points and at least $\lfloor y_1 \frac{q}{2} \rfloor$ good points.*

Proof. Given an infection sequence $\mathbf{v}$ let n_g be the number of its good points and observe that if $v^{(i)}$ is bad then $v_2^{(i+1)} < v_2^{(i)}$ and $v_1^{(i)} - v_1^{(i+1)} \le L$. Hence, $(n - n_g) \le y_2$ and

$$(n - n_g)L + n_g/q \ge y_1 - L,$$

i.e. $n_g \ge q(y_1 - L(1 + y_2))$. In particular, if $1 \le y_2 \le 2^{-\theta_q^2 \alpha} y_1$ then $n_g \ge \lfloor y_1 q/2 \rfloor$ for q small enough. $\square$

Given $\mathbf{v} \in \mathcal{S}(y)$ let $(w^{(1)}, w^{(2)}, \dots, w^{(n_y)})$ be the collection of the first $n_y := \lfloor y_1 \frac{q}{2} \rfloor$ good points of $\mathbf{v}$ ordered from the last one to the first one. By construction, $w^{(k-1)} \prec h(w^{(k)})$ $\forall k$. Using Definition 7.9, the event $\{\xi(y) = \mathbf{v}\}$ implies the event

$$G_{\mathbf{v}} := \cap_k \{\tau_{U_{w^{(k)}}} = \tau_{h(w^{(k)})}; \tau_{h(w^{(k)})} \ge \tau_{w^{(k-1)}}\},$$

and $\tau_y \geq \sum_k (\tau_{w^{(k)}} - \tau_{h(w^{(k)})})$. Therefore, $\forall \lambda > 0$ the definition of the event $G_{\mathbf{v}}$ together with a repeated use of the strong Markov property implies that

$$e^{-\lambda \mathbb{E}_{\omega_*}(\tau_y)} \leq \mathbb{E}_{\omega_*}(e^{-\lambda\tau_y}) \leq \sum_{\mathbf{v}\in\mathcal{S}(y)} \mathbb{E}_{\omega_*}\left(\mathbb{1}_{G_{\mathbf{v}}} e^{-\lambda\sum_{k=1}^{n_y}(\tau_{w^{(k)}} - \tau_{h(w^{(k)})})}\right) \tag{7.12}$$
$$\leq |\mathcal{S}(y)| \max_{\mathbf{v}} \mathbb{E}_{\omega_*}\Big(\mathbb{1}_{G_{\mathbf{v}}} \prod_{k=1}^{n_y} e^{-\lambda(\tau_{w^{(k)}} - \tau_{h(w^{(k)})})}\Big)$$
$$\leq |\mathcal{S}(y)| F(\lambda)^{n_y},$$

where $|\mathcal{S}(y)|$ denotes the cardinality of $\mathcal{S}(y)$ and

$$F(\lambda) := \max_{z\in\mathbb{Z}_+^2 :\, h(z)\in\mathbb{Z}_+^2} \ \max_{\omega:\, \omega(h(z))=0,\, \omega\restriction_{U_z}=1} \mathbb{E}_\omega\Big(e^{-\lambda\tau_z}\Big).$$

The next two lemmas provide the necessary bounds on $|\mathcal{S}(y)|$ and $F(\lambda)$.

Lemma 7.12. *For any $y \in \mathbb{Z}_+^2$ with $1 \leq y_2 < y_1 2^{-\alpha\theta_q^2}$ as $q \to 0$, we have*

$$|\mathcal{S}(y)| \leq \Big(y_1/y_2\Big)^{O(y_2)}.$$

Proof. Recall that a good point of an infection sequence specifies uniquely the next point of the sequence. Hence, we can reconstruct the full infection sequence by specifying which points are bad together with their relative position w.r.t. the previous point. Using Lemma 7.11 and Remark 7.10, it also follows that the length n of any infection sequence satisfies $n \in [n_y, q(y_1+y_2)]$. Thus for q small enough

$$|\mathcal{S}(y)| \leq \sum_{n=n_y}^{\lceil q(y_1+y_2)\rceil} \sum_{m=0}^{y_2} \binom{n}{m}(2L)^m \leq \sum_{n=n_y}^{\lceil q(y_1+y_2)\rceil} \binom{n}{y_2}(y_2+1)(2L)^{y_2}$$
$$\leq e^{O(\theta_q^2)y_2} \times O(q)y_1 \times \binom{\lceil q(y_1+y_2)\rceil}{y_2} \leq \Big(y_1/y_2\Big)^{O(y_2)}.$$

□

Lemma 7.13. *Fix $0 < \varepsilon \ll 1$ and let $T_\alpha = T_\alpha(\varepsilon, q) = 2^{\frac{\theta_q^2}{4}((1+4\alpha)\wedge 2)(1-2\varepsilon)}$. Then for any q sufficiently small and any $\lambda > 0$*

$$F(\lambda) \leq e^{-\lambda T_\alpha} + 2^{-\Omega(\varepsilon)\theta_q^2}.$$

Proof. Fix $z \in \mathbb{Z}_+^2$ such that $h(z) \in \mathbb{Z}_+^2$ together with ω such that $\omega(h(z)) = 0$ and $\omega\restriction_{U_z} = 1$. Let also $A := \{h(z) + \mathbf{e}_1 - \mathbf{e}_2, h(z) + 2\mathbf{e}_1 - \mathbf{e}_2, \ldots, z - \mathbf{e}_2\}$. Then,

$$\mathbb{E}_\omega(e^{-\lambda\tau_z}) \leq e^{-\lambda T_\alpha} + \mathbb{P}_\omega(\tau_z < T_\alpha)$$
$$\leq e^{-\lambda T_\alpha} + \mathbb{P}_\omega(\{\tau_z < T_\alpha\} \cap \{\tau_A > T_\alpha\}) + \mathbb{P}_\omega(\tau_A \leq T_\alpha)$$
$$\leq e^{-\lambda T_\alpha} + \mathbb{P}_\omega(\{\tau_z < T_\alpha\} \cap \{\tau_A > T_\alpha\}) + \sum_{a\in A} \mathbb{P}_\omega(\tau_a \leq T_\alpha).$$

Let $\mathcal{F}_{T_\alpha}$ be the σ-algebra generated by the variables $\omega_z(s), s \in [0, T_\alpha]$ where $z \in \{a \in \mathbb{Z}_+^2 : a \prec h(z)\} \cup \{a \in \mathbb{Z}_+^2 : a \prec b$ for some $b \in A\}$. Clearly $\{\tau_A > T_\alpha\} \in \mathcal{F}_{T_\alpha}$. Moreover, conditionally on $\mathcal{F}_{T_\alpha}$ and on the event $\{\tau_A > T_\alpha\}$, the East process on $A + \mathbf{e}_2$ coincides up to time T_α with the one-dimensional

East chain on $A + \mathbf{e}_2$ with a boundary value at $\{\omega_{h(w)}(s)\}_{s\le T}$ which is measurable w.r.t. $\mathcal{F}_{T_\alpha}$. We can then apply Corollary 7.5 with $d = 1$ and $n = \lfloor \theta_q \rfloor$ to obtain:

$$\mathbb{P}_\omega(\{\tau_z < T_\alpha\} \cap \{\tau_A > T_\alpha\}) \le O(T_\alpha)2^{-\frac{\theta_q^2}{2}(1-\varepsilon)} = O\Big(2^{-\frac{\theta_q^2}{4}((2-(1+4\alpha)\wedge 2)(1-2\varepsilon)+2\varepsilon)}\Big) \le 2^{-\Omega(\varepsilon)\theta_q^2}.$$

Let $n_A = \min_{a\in A}\min_{z'\prec a,\, z'\notin U_z}\|a - z'\|_1$, and observe that $\exists\, \varepsilon(\alpha) > 0$ such that $\forall\, \varepsilon \le \varepsilon(\alpha)$ and all q small enough depending on ε, $T_\alpha \le n_A\, 2^{\frac{\theta_q^2}{4}(1-\varepsilon)}$. We can then use Remark 7.7 to get that

$$\sum_{a\in A} \max_{\omega:\ \omega\restriction_{U_z}=1} \mathbb{P}_\omega(\tau_a \le T_\alpha) \le e^{-\Omega(\varepsilon\theta_q^2\lfloor n_A 2^{-\frac{\theta_q}{2}}\rfloor)} \le 2^{-\Omega(\varepsilon)\theta_q^2},$$

because $n_A \ge L - 2^{\theta_q} \gg 2^{\theta_q/2}$. □

We can now conclude the proof. By combining the two lemmas above and choosing $\lambda = \lambda_\alpha(q) = T_\alpha^{-1}\varepsilon\theta_q^2$, we get from Equation (7.12) that

$$e^{-\lambda \mathbb{E}_{\omega^*}(\tau_y)} \le |\mathcal{S}(y)| F(\lambda)^{n_y} \le \Big(y_1/y_2\Big)^{O(y_2)} e^{-\Omega(\varepsilon)\theta_q^2 n_y},$$

where we recall that $n_y := \lfloor y_1 \frac{q}{2} \rfloor$. If $y = \lfloor n\mathbf{x} \rfloor$ with $\mathbf{x}$ such that $0 < x_2 \le x_1 2^{-\theta_q^2\alpha}$, the above inequality implies

$$\mathbb{E}_{\omega^*}(\tau_{\lfloor n\mathbf{x}\rfloor}) \ge \Omega(q\, T_\alpha) \times n \quad \text{as } n \to \infty.$$

In particular $v_{\max}(\mathbf{x}) \le 2^{-\frac{\theta_q^2}{4}((1+4\alpha)\wedge 2)(1-o(1))}$.

7.3 Mixing behind the front: Proof of Theorem 3

Let us now come to the proof that for $t \to \infty$, $q \to 0$ and $\varepsilon \to 0$ we find equilibrium in

$$\Lambda(\delta, \varepsilon, t) = \{x \in \mathbb{Z}_+^d : \min_{i,j} x_i/x_j \ge \delta \text{ and } \|x\|_1 \le 2^{-\frac{\theta_q^2}{2d}(1+\varepsilon)} \times t\},$$

if $\delta > 0$ and do not if $\delta = 0$. We begin with the case $\delta = 0$.

Recall Remark 2.1 and that $v_{\min}(\mathbf{e}_i) = v_{\max}(\mathbf{e}_i) = 2^{-\frac{\theta_q^2}{2}(1+o(1))}\ \forall\, i \in [d]$. Take $0 < \varepsilon \ll 1$ and let $x_t = \lfloor 2^{-\frac{\theta_q^2}{2d}(1+\varepsilon)}\, t \rfloor\, \mathbf{e}_1,\ t \gg 0$. By construction $x_t \in \Lambda(\delta = 0, \varepsilon, t)$. Let also

$$A_t = \{\omega :\ \exists\, y \in \{x_t - \lfloor 2^{2\theta_q} \rfloor \mathbf{e}_1, \ldots, x_t\} \text{ such that } \omega_y(t) = 0\},$$

and use

$$\|\nu_t^{\delta,\varepsilon} - \mu_{\Lambda(\delta,\varepsilon,t)}\|_{TV} \ge |\mu(A_t) - \nu_t^{\delta,\varepsilon}(A_t)|.$$

For any t large enough $\mu(A_t) = 1 - e^{-\Omega(2^{\theta_q})}$, while Remark 7.7 gives $\limsup_{t\to\infty} \nu_t^{\delta,\varepsilon}(A_t) = 0$. Hence,

$$\liminf_{q\to 0} \liminf_{t\to\infty} \|\nu_t^{\delta,\varepsilon} - \mu_{\Lambda(\delta,\varepsilon,t)}\|_{TV} = 1.$$

Next we consider the case $0 < \delta < 1$. Fix $0 < \varepsilon \ll 1$ and let us begin by following [15]. The first observation (see [15, Lemma 5.5]) is that equilibrium in the region $\Lambda(\delta, \varepsilon, t)$ is achieved very rapidly, within a time $O(\log(|\Lambda(\delta, \varepsilon, t)|)^{4d})$, if the initial configuration has a vacancy in every interval of $\Lambda(\delta,\varepsilon,t)$ parallel to a coordinate direction and containing $O(\log(|\Lambda(\delta,\varepsilon,t)|)^2)$ vertices. Hence, if the above condition is satisfied by the East process at time $t/2$ then at time t the measure $\nu_t^{\delta,\varepsilon}$ will be very close to $\mu_{\Lambda(\delta,\varepsilon,t)}$ in the total variation distance. The second observation (cf. [15, Lemma 5.3]) is the following. Recall that τ_x is the first time a vacancy appears at x. Then the above requirement for the East process at time $t/2$ will be satisfied with w.h.p. if $\tau_x \le t/2 - O(\log(|\Lambda(\delta,\varepsilon,t)|)^2)\ \forall\, x \in \Lambda(\delta,\varepsilon,t)$. A more precise formulation of the above two steps is as follows.

Lemma 7.14. *For any t large enough depending on q,δ,ε*

$$\|\mu_{\Lambda(\delta,\varepsilon,t)} - \nu_t^{\delta,\varepsilon}\|_{TV} \le \varepsilon + \sum_{x\in\Lambda(\delta,\varepsilon,t)} \mathbb{P}_{\omega^*}(\tau_x > t/3). \tag{7.13}$$

We decided to skip the proof of the lemma as it follows very closely the proofs of Lemma 5.3 and 5.5 of [15].

The proof of the theorem then boils down to proving that the second term in the r.h.s. of (7.13) vanishes as $t \to \infty$. For future needs we actually prove a slightly stronger result.

Lemma 7.15.

$$\limsup_{t\to\infty} \sup_{y\in\mathbb{Z}_+^d} \sum_{x\in\Lambda(\delta,\varepsilon,t)+y} \sup_{\omega:\, c_y(\omega)=1} \mathbb{P}_\omega(\tau_x > t/3) = 0. \tag{7.14}$$

Proof of the lemma. Fix $y \in \mathbb{Z}_+^d$ together with ω such that $c_y(\omega) = 1$. All the estimates in the sequel will be uniform in y,ω. Fix $x \in \Lambda(\delta,\varepsilon,t) + y$ and let $\mathbf{x} = (x-y)/|x-y|$ be the associated unit vector in $\mathbb{R}_+^d$. Clearly $\min_{i,j} \mathbf{x}_i/\mathbf{x}_j \ge \delta$. Let $\ell = 2^{\theta_q^{3/2}}$, $n_x = \lfloor |x-y|/\ell \rfloor$, and set $x^{(0)} = y$, $x^{(n)} = \lfloor n\ell\mathbf{x}\rfloor, n \in [n_x]$, $x^{(n_x+1)} = x$. By construction $|x^{(n+1)} - x^{(n)}| \le \ell + 1$, and $\exists\, \kappa(\delta) \ge 1, q(\delta) < 1$ such that $\forall\, q \le q(\delta)$

$$\max_{0\le n\le n_x} \max_{i,j} \frac{(x^{(n+1)} - x^{(n)})_i}{(x^{(n+1)} - x^{(n)})_j} \le \kappa(\delta).$$

For the East process with initial condition ω recursively define

$$\tau^{(0)} = \inf\{s \ge 0, \omega_{x^{(0)}}(s) = 0\}, \quad \tau^{(n)} = \inf\{s \ge \tau^{(n-1)} :\ \omega_{x^{(n)}}(s) = 0\},$$

and set $\Delta_n = \tau^{(n)} - \tau^{(n-1)}$. Finally, let $M = \log(t)^{5d} \times 2^{\frac{\theta_q^2}{2d}(1+\varepsilon/2)}$. Using $\tau_x \le \sum_{n=1}^{n_x+1} \Delta_n$ we write

$$\mathbb{P}_\omega\Big(\tau_x \ge t/3\Big) \le \mathbb{P}_\omega\Big(\sum_{n=1}^{n_x+1} \Delta_n \mathbb{1}_{\{\Delta_n \le M\}} \ge t/3\Big) + \sum_{n=1}^{n_x+1} \sup_{\omega:\, \omega_{x^{(n-1)}}=0} \mathbb{P}_\omega\Big(\Delta_n \ge M\Big). \tag{7.15}$$

In order to bound from above the second term in Equation (7.15) we use Equation (7.6) with $\ell_t = \log^2(t)$ and $\max_i L_i = \ell$ to get

$$\sup_{\omega:\, \omega_{x^{(n-1)}}=0} \mathbb{P}_\omega\Big(\Delta_n \ge M\Big) \le t\ell_t^d e^{-cq\ell_t} + 2^{\theta_q(\ell_t+\ell)^d - M\ell_t^{-d} 2^{-\frac{\theta_q^2}{2}(1+\varepsilon)}}.$$

Hence, for any t large enough depending on q, the second term in the r.h.s. of (7.15) satisfies

$$\sum_{n=1}^{n_x+1} \sup_{\omega:\, \omega_{x^{(n-1)}}=0} \mathbb{P}_\omega\Big(\Delta_n \ge M\Big) \le e^{-\Omega(q\log^2(t))}.$$

We tackle the first term in the r.h.s. of (7.15) via the exponential Chebyshev inequality with $\lambda = 2^{-\frac{\theta_q^2}{2d}(1+\varepsilon/2)}\log^2(t)/t$. Using the strong Markov property and $\lambda M \le 1$ for any large enough t we obtain

$$\begin{aligned}\mathbb{P}_\omega\Big(\sum_{n=1}^{n_x+1}\Delta_n\mathbb{1}_{\{\Delta_n\le M\}}\ge t/3\Big) &\le e^{-\lambda t/3}\times\mathbb{E}_\omega\Big(\prod_{n=1}^{n_x+1}e^{\lambda\Delta_n\mathbb{1}_{\{\Delta_n\le M\}}}\Big)\\ &\le e^{-\lambda t/3}\times\Big(\sup_n\sup_{\omega:\,\omega_{x(n-1)}=0}\mathbb{E}_\omega\Big(e^{\lambda\Delta_n\mathbb{1}_{\{\Delta_n\le M\}}}\Big)\Big)^{n_x+1}\\ &\le e^{-\lambda t/3}\times\Big(1+e\lambda\sup_n\sup_{\omega:\,\omega_{x(n-1)}=0}\mathbb{E}_\omega\Big(\Delta_n\Big)\Big)^{n_x+1},\end{aligned}$$

where we used $e^a \le 1+ea$, $\forall\, 0\le a\le 1$ in the last inequality. We can finally appeal to Lemma 7.1 to get that for all q small enough depending on δ,ε

$$1+e\lambda\sup_n\sup_{\omega:\,\omega_{x(n-1)}=0}\mathbb{E}_\omega\Big(\Delta_n\Big)\le 1+e\lambda\,2^{(1+\varepsilon/2)\frac{\theta_q^2}{2d}}\le e^{e\log^2(t)/t}.$$

In conclusion,

$$\mathbb{P}_\omega\Big(\sum_{n=1}^{n_x+1}\Delta_n\mathbb{1}_{\{\Delta_n\le M\}}\ge t/3\Big)\le e^{-\lambda t/3+e(n_x+1)\log^2(t)/t}\le e^{-\lambda t/6},$$

where we used $(n_x+1)\le |x-y|+1\le t\,2^{-\frac{\theta_q^2}{2d}(1+\varepsilon)}+1$ to obtain the last inequality for q small enough depending on ε.

Summarizing, we have proved that $\exists\, q(\delta,\varepsilon)>0$ such that for any $q\le q(\delta,\varepsilon)$ and all t large enough

$$\sum_{x\in\Lambda(\delta,\varepsilon,t)+y}\mathbb{P}_\omega\Big(\tau_x>t/3\Big)\le e^{-\Omega\Big(2^{-(1+\varepsilon/2)\frac{\theta_q^2}{2d}}\log^2(t)\Big)}\tag{7.16}$$

and (7.14) follows. □

7.4 Cutoff phenomenon: Proof of Theorem 4

Using Remark 3.1 $d(t)\ge\bar d(t)$, where $\bar d(t)$ is defined as $d(t)$ but for the *one dimensional* East chain on $\{0,\dots,n\}$. Hence Equation (3.4) follows directly from the cutoff result for the latter chain (see Theorem 3.5). We now turn to the proof of Equation (3.5).

Let $w_n = n^{2/3}$ and let $\hat T_n = T_n + w_n/2$. As in the proof of Theorem 3 (see Equation (7.13)) the following can be proved by following very closely the proof of Lemma 5.3 and Lemma 5.5 of [15].

Lemma 7.16. *For any* $q\in(0,1)$

$$\limsup_{n\to\infty} d(T_n+w_n)\le\limsup_{n\to\infty}\max_{\omega\in\Omega_{\Lambda_n}}\mathbb{P}_\omega(\exists\, x\in\Lambda_n:\ \tau_x\ge\hat T_n).\tag{7.17}$$

We will prove that for q small enough the r.h.s. of Equation (7.17) is zero. In the sequel ε will be a small positive constant and q will be assumed to be sufficiently small, depending on ε. Using the symmetry of the East chain w.r.t. the line $x_1=x_2$ and the union bound, it is enough to prove that

$$\limsup_{n\to\infty}\max_{\omega\in\Omega_{\Lambda_n}}\sum_{x\in\bar\Lambda_n}\mathbb{P}_\omega(\tau_x\ge\hat T_n)=0,\tag{7.18}$$

where $\bar{\Lambda}_n = \{x \in \Lambda_n : x_1 \geq x_2\}$.

The intuition behind Equation (7.18) is as follows. For any $x \in \bar{\Lambda}_n$ the infection time τ_x should be dominated by the infection time of the vertex $(x_1 - x_2, 0)$ plus the infection time of x given that the chain starts with a vacancy at $(x_1 - x_2, 0)$. Using Theorem 3.5 the first time is, with great accuracy, $(x_1 - x_2)/v$, while part (A) of Theorem 1 suggests that w.h.p. the second time is $O\Big(x_2/v_{\min}(\hat{\mathbf{e}})\Big)$ where $\hat{\mathbf{e}} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. Hence, we expect τ_x to satisfy w.h.p.

$$\tau_x \lesssim (x_1 - x_2)/v + O\Big(x_2/v_{\min}(\hat{\mathbf{e}})\Big) \lesssim n/v \quad \forall x \in \bar{\Lambda}_n,$$

because $v_{\min}(\hat{\mathbf{e}}) \gg v$. In other words, the time needed to infect all vertices of $\bar{\Lambda}_n$ should be dominated by the time needed to infect all vertices on the horizontal side of Λ_n. In turn, using the one dimensional cutoff result the latter time is smaller than $\hat{T}_n$ w.h.p.

We will now detail the intuition above. We cover $\bar{\Lambda}_n$ with three regions:

$$\begin{aligned}
\Lambda_n^{(1)} &= \{x \in \bar{\Lambda}_n : x_2 \geq x_1/3\}, \\
\Lambda_n^{(2)} &= \{x \in \bar{\Lambda}_n : x_2 \leq \log(n)^4\}, \\
\Lambda_n^{(3)} &= \{x \in \bar{\Lambda}_n : \log(n)^4 \leq x_2 \leq x_1/3\},
\end{aligned}$$

and we will prove that

$$\limsup_{n\to\infty} \max_{\omega \in \Omega_{\Lambda_n}} \sum_{x \in \Lambda_n^{(i)}} \mathbb{P}_\omega(\tau_x \geq \hat{T}_n) = 0, \ \forall\, i \in [3]. \tag{7.19}$$

i = 1. In this case Equation (7.19) follows from Equation (7.14) together with the observation that $\Lambda_n^{(1)} \subset \Lambda(\delta, \varepsilon, t)$ if $\delta = 1/3$ and $t = 2^{\frac{\theta_q^2}{4}(1+\varepsilon)} 2n$, and that $\hat{T}_n \gg t/3$.

i = 2. Fix $x \in \Lambda_n^{(2)}$ and write $\hat{x}$ for the vertex $(x_1, 0)$. Then, $\tau_x \leq \tau_{\hat{x}} + \sigma_x$, where $\sigma_x = \inf\{s \geq \hat{\tau}_x : \omega_x(s) = 0\}$. Using the strong Markov property we get

$$\max_\omega \mathbb{P}_\omega(\tau_x \geq \hat{T}_n) \leq \max_\omega \mathbb{P}_\omega(\tau_{\hat{x}} \geq \hat{T}_n - w_n/4) + \max_{\omega\colon \omega_{\hat{x}}=0} \mathbb{P}_\omega(\tau_x > w_n/4).$$

Using once again Theorem 3.5

$$\limsup_{n\to\infty} \sum_{x \in \Lambda_n^{(2)}} \max_\omega \mathbb{P}_\omega(\tau_{\hat{x}} \geq \hat{T}_n - w_n/4) = 0.$$

Notice that $\|x - \hat{x}\|_1 = x_2 \leq \log(n)^4 \ll w_n/4$. Hence, the term $\max_{\omega\colon \omega_{\hat{x}}=0} \mathbb{P}_\omega(\tau_x > w_n/4)$ can be bounded from above exactly as in the derivation of Equation (7.6) with parameter $t = w_n/4$. The final result is

$$\max_{\omega\colon \omega_{\hat{x}}=0} \mathbb{P}_\omega(\tau_x > w_n/4) \leq e^{-c(q) w_n^{1/8}}, \quad c(q) > 0,$$

so that

$$\limsup_{n\to\infty} \sum_{x \in \Lambda_n^{(2)}} \max_{\omega\colon \omega_{\hat{x}}=0} \mathbb{P}_\omega(\tau_x > w_n/4) = 0.$$

$\mathbf{i=3}$. For any $x \in \Lambda_n^{(3)}$ let $\phi(x) = x_1 - x_2$ and set $\hat{x} = (\phi(x), 0)$. By construction, the direction of the vector $x - \hat{x}$ is the $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$-direction and $2\log(n)^4 \leq 2x_2 \leq \phi(x) \leq n - \log(n)^4$. As in the previous step we write $\tau_x \leq \tau_{\hat{x}} + \sigma_x$ to get

$$\max_{\omega} \mathbb{P}_\omega\Big(\tau_x \geq \hat{T}_n\Big) \leq \max_{\omega} \mathbb{P}_\omega\Big(\tau_{\hat{x}} \geq \frac{\phi(x)}{v} + \frac{\phi(x)^{2/3}}{4}\Big) + \max_{\omega:\,\omega_{\hat{x}}=0} \mathbb{P}_\omega\Big(\tau_x > \hat{T}_n - (\frac{\phi(x)}{v} + \frac{\phi(x)^{2/3}}{4})\Big). \tag{7.20}$$

Using Theorem 3.5 applied to the interval $\{0, \dots, \phi(x)\}$ together with $\phi(x) \geq 2\log(n)^4$, we get that the first term in the r.h.s. of Equation (7.20) is bounded from above by $e^{-c(q)\phi(x)^{1/3}} \leq e^{-c'(q)\log(n)^{4/3}}$ for large n, so that

$$\limsup_{n\to\infty} \sum_{x \in \Lambda_n^{(3)}} \max_{\omega} \mathbb{P}_\omega\Big(\tau_{\hat{x}} \geq \frac{\phi(x)}{v} + \phi(x)^{2/3}\Big) = 0.$$

We finally deal with the second term in the r.h.s. of Equation (7.20). Here the key observation is that

$$\hat{T}_n - (\frac{\phi(x)}{v} + \frac{\phi(x)^{2/3}}{4}) \geq \frac{w_n}{2} + \frac{x_2}{v} \gg 2^{\frac{\theta_q^2}{4}(1+\varepsilon)} x_2,$$

because $v = 2^{-\frac{\theta_q^2}{2}(1+o(1))}$. Hence we can apply Equation (7.16) with

$$y = \hat{x} + \mathbf{e}_1, \quad \delta = \frac{1}{3}, \quad t = 3\Big(\hat{T}_n - (\frac{\phi(x)}{v} + \frac{\phi(x)^{2/3}}{4})\Big)$$

to get that

$$\max_{\omega:\,\omega_{\hat{x}}=0} \mathbb{P}_\omega\Big(\tau_x > \hat{T}_n - (\frac{\phi(x)}{v} + \frac{\phi(x)^{2/3}}{4})\Big) \leq e^{-\Omega\Big(2^{-(1+\varepsilon/2)\frac{\theta_q^2}{4}} \log^2(\hat{T}_n - \frac{\phi(x)}{v} - \frac{\phi(x)^{2/3}}{4})\Big)} \leq e^{-c_q \log(w_n)^2}.$$

In conclusion,

$$\limsup_{n\to\infty} \sum_{x \in \Lambda_n^{(3)}} \max_{\omega:\,\omega_{\hat{x}}=0} \mathbb{P}_\omega\Big(\tau_x > \hat{T}_n - (\frac{\phi(x)}{v} + \frac{\phi(x)^{2/3}}{4})\Big) = 0.$$

$\square$

Chapter 8

MCEM Ergodicity result: Proof of Theorem 5

In this chapter we prove the non-ergodicity of the G-MCEM if $G = H_d$ and positivity of the spectral gap in the two cases outlined in Theorem 5(B). Non-ergodicity for $G = H_d$ follows from a geometric argument by defining a state on H_d for which there is no legal transition possible. To find positivity of the spectral gap we first require four tools presented in Section 8.1.

The first is the exterior condition theorem in Section 8.1.1 that gives us a Poincaré-inequality of auxiliary models with constraints that have low failure probability and satisfy a geometric condition. Then in Section 8.1.2 we prove that the spectral gap is monotone in G. Using this, we identify configurations on boxes Λ that allow for legal paths of finite length that flip vertices in $\Lambda + \mathbf{v}$ for some vector $\mathbf{v}$. The final two tools are small mathematical Lemmas that find use throughout the thesis.

We start with the proof of part (A).

Proof of Theorem 5(A). If $G = H_d$ say that $\omega \in \Omega_{H_d}$ is in a blocked state if $\omega_{\mathbf{1}-h} = h$ for each $h \in H_d$. By construction, there is no legal transition out of a blocked state since to transition the h-vacancy at $\mathbf{1} - h$ to $\star$ you need another h-vacancy inside H_d but every vertex in H_d is already occupied by a different vacancy type. Say that $\omega \in \mathcal{A}$ if $\omega \restriction_{H_d}$ is in a blocked state. Then $\mathbb{1}_{\mathcal{A}}$ is not a constant function but $\mathcal{D}(\mathbb{1}_{\mathcal{A}}) = 0$ while $\mu(\mathcal{A}) > 0$ so that $\mathrm{Var}_\mu(\mathbb{1}_{\mathcal{A}}) > 0$ and so by Theorem 2.2 we get the claim. □

8.1 Four key tools

Before coming to the finiteness of the spectral gap we need to introduce some key tools.

8.1.1 A constrained Poincaré inequality for product measures

Let us recall the notion of *exterior conditions* and a Poincaré inequality based on it from [44, Section 2.3 and 2.4]. We state it here again without proof as it is one of the main ingredients to bound the spectral gap in the following chapters. We define the support $\mathrm{Supp}(\mathcal{A})$ of an event $\mathcal{A}$ as the set of vertices the event depends on. As with all chapters on the G-MCEM given a $G \subset H_d$, the associated state space is $\Omega = \mathcal{S}(G)^{\mathbb{Z}^d}$ and we assume there to be a valid parameter set $\mathbf{q}$ and the equilibrium measure μ without explicitly specifying it every time.

Definition 8.1 (Exterior condition)**.** Given an increasing and exhausting collection of subsets $\{V_n\}_{n\in\mathbb{Z}}$ of $\mathbb{Z}^d$ (i.e. $V_n \subset V_{n+1}$ for all n and $\cup_n V_n = \mathbb{Z}^d$), let the exterior of $x \in V_n$ be the set $\mathrm{Ext}_x :=$

$\cup_{j=n}^{\infty} V_{j+1} \setminus V_j$. We then say that the family of events $\{\mathcal{A}_x\}_{x\in\mathbb{Z}^d}$ *satisfies the exterior condition* w.r.t. $\{V_n\}_{n\in\mathbb{Z}}$ if $\mathrm{Supp}(\mathcal{A}_x) \subset \mathrm{Ext}_x$ for all $x \in \mathbb{Z}^d$.

Let $\{\mathcal{A}_x^{(i)}\}_{x\in\mathbb{Z}^d}, i = 1, \dots, k$ be a family of events and for any nonempty $I \subset [k]$ let $\mathrm{Supp}(\mathcal{A}_x^{(I)}) = \bigcup_{i\in I} \mathrm{Supp}(\mathcal{A}_x^{(i)})$.

Theorem 8.2 (Exterior condition theorem, [44, Theorem 2])**.** *Assume that*

$$(2^k - 1) \sup_{z\in\mathbb{Z}^2} \sum_{\substack{J\subset[k]\\ J\neq\emptyset}} \sum_{\substack{x\in\mathbb{Z}^2\\ \{x\}\cup\mathrm{Supp}(\mathcal{A}_x^{(J)})\ni z}} \mu\left(\prod_{i\in J}(1 - \mathbb{1}_{\mathcal{A}_x^{(j)}})\right) < 1/4. \tag{8.1}$$

Suppose in addition that there exists an exhausting and increasing family $\{V_n\}_{n\in\mathbb{Z}}$ of subsets of $\mathbb{Z}^d$ such that, for any $i \in [k]$, the events $\{\mathcal{A}_x^{(i)}\}_{x\in\mathbb{Z}^d}$ satisfy the exterior condition w.r.t. $\{V_n\}_{n\in\mathbb{Z}}$. Then, for any local function $f : \Omega \to \mathbb{R}$ we have

$$\mathrm{Var}(f) \leq 4 \sum_x \mu\left(\left[\prod_{j=1}^{k} \mathbb{1}_{\mathcal{A}_x^{(j)}}\right] \mathrm{Var}_x(f)\right). \tag{8.2}$$

In particular, the same conclusion holds if, instead of Equation (8.1) *we have that*

$$\lim_{q_{\min}\to 0} \max_{j\in[k]} \left[\sup_{x\in\mathbb{Z}^2} |\mathrm{Supp}(\mathcal{A}_x^{(j)})| \sup_{x\in\mathbb{Z}^2} \mu\Big(1 - \mathbb{1}_{\mathcal{A}_x^{(j)}}\Big)\right] = 0 \tag{8.3}$$

Proof. Equation (8.3) implies Equation (8.1) and the proof how Equation (8.1) implies Equation (8.2) is in [44]. Note that [44] made the statement with KCM in mind, but the proof only uses that μ is a product measure so applies equally to MCEM. □

Remark 8.3. We added Equation (8.3) as in the proof of Theorem 6 we always take $q_{\min} \to 0$ and this condition is more straightforward to check since we can analyse the families $\{\mathcal{A}_x^{(i)}\}_{x\in\mathbb{Z}^d}$ for the various $i \in [k]$ independently from each other.

8.1.2 Monotonicity in G of the spectral gap

Naturally one conjectures that the more vacancy types are added to the G-MCEM the lower the spectral gap should be as the model gets progressively more jammed through the interaction of the various vacancy types. Indeed, the next result shows this is the case.

Lemma 8.4. *For any $G' \subset G \subset H_d$ and valid parameter set $\mathbf{q}$ for the G-MCEM we have*

$$\gamma(G, \mathbf{q}) \leq \gamma(G', \mathbf{q}')$$

with $\mathbf{q}' = \{q_h : h \in G'\}$ and in particular

$$\gamma(G, \mathbf{q}) \leq \gamma_d(q_{\min}).$$

Proof. Let $G' \subset G \subset H_d$ and fix a parameter set $\mathbf{q}$ for the G-MCEM. Recall that $\mathcal{S}(G) = G \cup \{\star\}$. Define the projection φ on $\mathcal{S}(G)$ to $\mathcal{S}(G')$ that maps G' onto itself and $\mathcal{S}(G) \setminus G'$ to $\star$. We then have, through the variational characterisation of the spectral gap Equation (4.2), that

$$\gamma(G, \mathbf{q}) = \inf_{\substack{f\in\mathrm{Dom}(\mathcal{L}^{(G,\mathbf{q})})\\ f\neq\mathrm{const}}} \frac{\mathcal{D}(f)}{\mathrm{Var}(f)} \leq \inf_{\substack{g\in\mathrm{Dom}(\mathcal{L}^{(G',\mathbf{q})})\\ g\neq\mathrm{const}}} \frac{\mathcal{D}(g\circ\varphi)}{\mathrm{Var}(g\circ\varphi)},$$

where we write $\mathcal{L}^{(G,\mathbf{q})}$ for the generator of the G-MCEM to make the G-dependence explicit in this proof. Write ν' for the measure on $\mathcal{S}(G')$ that assigns probability q_h to $h \in G'$ and $p' := 1 - \sum_{h \in G'} q_h$ to $\star$ and let μ' be the product measure of ν'. Since $\mu(g \circ \varphi) = \mu'(g)$ and $(g \circ \varphi)(\cdot)^2 = g^2 \circ \varphi(\cdot)$ we get $\mathrm{Var}(g \circ \varphi) = \mathrm{Var}_{\mu'}(g)$. For the Dirichlet form we get (recall Equation (4.1))

$$\begin{aligned}\mathcal{D}(g \circ \varphi) &= \sum_{h \in G} \sum_{x \in \mathbb{Z}^2} \mu\Big[c_x^h q_h p\big(\nabla_x^{(h)}(g \circ \varphi)\big)^2\Big] \\ &\leq \sum_{h \in G'} \sum_{x \in \mathbb{Z}^2} \mu'\Big[c_x^h q_h p'\big(\nabla_x^{(h)}(g)\big)^2\Big]\end{aligned}$$

where we used that the constraints c_x^h only check whether a qualified neighbour is h or not, and thus is the same for the G-MCEM and the G'-MCEM if $h \in G'$. Further $\nabla_x^{(h)}(g \circ \varphi)(\omega) = 0$ if $\omega_x \notin G'$ and so we could replace μ with μ'. The r.h.s. is equal to the Dirichlet form of the G'-MCEM with parameter set $\mathbf{q}$ so we get the first part of the claim.

The second part follows analogously by mapping the h with the lowest equilibrium density to 0 and all the other states to 1 thus recovering the spectral gap $\gamma_d(q_{\min})$ of the East model with vacancy density $q_{\min}$. □

8.1.3 Variance as transition terms and the path method

Given the valid parameter set $\mathbf{q}$, recall the measure ν, defined in Section 4.2 as the measure on $\mathcal{S}(G) = \{\star\} \cup G$ that assigns probability q_h to $h \in G$ and p to $\star$.

Lemma 8.5 (Variance as transition terms). *For any function $f : \mathcal{S}(G) \to \mathbb{R}$ we find*

$$\mathrm{Var}_\nu(f)(\omega) \leq 2 \sum_{h \in G} q_h \big(\nabla^{(h)}(f)\big)^2(\omega). \tag{8.4}$$

Proof. Writing $p = q_\star$ in this proof we have

$$\begin{aligned}\frac{1}{2} \sum_{\omega, \omega' \in S(G)} q_\omega q_{\omega'} \big(f(\omega) - f(\omega')\big)^2 &= \sum_{\omega \in S(G)} q_\omega f(\omega)^2 - \left(\sum_{\omega \in S(G)} q_\omega f(\omega)\right)^2 \\ &= \mathrm{Var}_\nu(f).\end{aligned}$$

Applying Cauchy-Schwarz gives

$$\big(f(\omega) - f(\omega')\big)^2 \leq 2\Big(\big(f(\omega) - f(\star)\big)^2 + \big(f(\star) - f(\omega')\big)^2\Big)$$

and thus the claim. □

Remark 8.6. Recall the discussion after Theorem 6 about the case $p \to 0$ and why we made the assumption that $p \geq \Delta > 0$. We will often revwrite variances over sets as transition terms using Lemma 8.5. Instead of the form presented here we often need a term corresponding to the Dirichlet form of an East process on the r.h.s. in Equation (8.4). For this, we are missing a p, so that the r.h.s. can be rewritten to $2/p\mathcal{D}(f)$. This means that in the final estimate we have at least a contribution of $1/p$, and often we apply the above Lemma twice in the same proof leading to a contribution of $1/p^2$ which diverges as $p \to 0$ so it would need to be included in Theorem 6 if we did not make the assumption that $p > \Delta$ giving an estimate of the spectral gap between $\gamma_2(q_{\min})$ and $\gamma_2(q_{\min})/p^2$. This contribution can be interpreted as the necessary waiting time for a $\star$-ring (recall the graphical construction), since $1/p$ is the expectation of the geometric variable with success probability p. While there has to be a contribution in $\gamma(G, \mathbf{q})$ that diverges with $p \to 0$ if $|G| \geq 2$ (if we never go to the neutral state, the colours can never change), in the absence of a fitting lower bound we decided that the more interesting part was showing that the dynamics are dominated by the two-dimensional East dynamics with parameter $q_{\min}$ in the cases (2.i),(2.ii) or (3.i)-(3.iii) if $p > \Delta$.

Analogously to Definition 7.2, say that a family of configurations $\{(\omega^{(i)})\}_{i\in[n]}$ is a *legal path* if each transition from $\omega^{(i)}$ to $\omega^{(i+1)}$ is legal for the G-MCEM, where the specific G will be clear from context. Recall further that $x \prec^{(h)} y$ for $h \in H_d$ if $x \cdot \mathbf{v} \leq y \cdot \mathbf{v}$ for any $\mathbf{v} \in \mathcal{P}(h)$.

Our second tool, the path method, is a well known trick in estimating the spectral gap see for example [11, Proposition 6.6] or [29] for uses in other contexts. Recall for this the notation of $\mathcal{D}_\Lambda$ introduced in Remark 4.6 where the sum over all vertices in $\mathbb{Z}^d$ is replaced by the sum over $\Lambda \subset \mathbb{Z}^d$ and the equilibrium measure μ by μ_Λ.

Lemma 8.7 (The path method)**.** *Let $\omega, \eta \in \Omega$ and let $\Gamma = (\omega^{(1)}, \ldots, \omega^{(n)})$ be a legal path such that $\omega^{(1)} = \omega$ and $\omega^{(n)} = \eta$ and let $\Lambda \subset \mathbb{Z}^d$ consist of those vertices x such that $\omega_x^{(i)} \neq \omega_x^{(i+1)}$ for some $i \in [n]$. Then, for any $f : \Omega \to \mathbb{R}$*

$$\mu_\Lambda(\omega)(f(\omega) - f(\eta))^2 \leq \frac{n}{\min(\mathbf{q}, p)} \max_{i\in[n]} \frac{\mu_\Lambda(\omega)}{\mu_\Lambda(\omega^{(i)})} \mathcal{D}_\Lambda(f).$$

Proof. Write $f(\omega) - f(\eta) = \sum_{i\in[n-1]} f(\omega^{(i)}) - f(\omega^{(i+1)})$ as a telescopic sum and use Cauchy-Schwarz to get

$$\begin{aligned}
\mu_\Lambda(\omega)(f(\omega) - f(\eta))^2 &\leq n \sum_{i\in[n-1]} \mu_\Lambda(\omega)\big(f(\omega^{(i)}) - f(\omega^{(i+1)})\big)^2 \\
&\leq n \max_{i\in[n]} \frac{\mu_\Lambda(\omega)}{\mu_\Lambda(\omega^{(i)})} \sum_{i\in[n-1]} \mu_\Lambda(\omega^{(i)})\big(f(\omega^{(i)}) - f(\omega^{(i+1)})\big)^2 \\
&\leq \frac{n}{\min(\mathbf{q}, p)} \max_{i\in[n]} \frac{\mu_\Lambda(\omega)}{\mu_\Lambda(\omega^{(i)})} \mathcal{D}_\Lambda(f),
\end{aligned}$$

where in the last inequality we used that for $\omega^{(i)} \to \omega^{(i+1)}$ to be a legal transition there is exactly one x such that $\omega_x^{(i)} \neq \omega_x^{(i+1)}$. Assume without loss of generality that $\omega_x^{(i)} = \star$ and $\omega_x^{(i+1)} = h$ for $h \in G$ then we have to extend by q_h/q_h which results in the Dirichlet form plus the extra term q_h which we estimate by $\min(\mathbf{q}, p)$. ☐

In the proofs of part (B) and (C) of Theorem 5 we do not explicitly mention the length of the involved paths as the important thing is that they are finite not how they scale. In Theorem 6 instead it is of crucial importance to know the exact scaling.

8.2 Vacancies with a common direction: Proof of Theorem 5(B.i)

In this section we present the proof of part (B.i) in which G is such that all $h \in G$ share a propagation direction.

Using Lemma 8.4, w.l.o.g. we can assume that $G = \{h_\mathbf{j} \colon \mathbf{j} \in \{0,1\}^{d-1} \simeq H_{d-1}\}$ where $h_\mathbf{j} = (\mathbf{j}_1, \ldots, \mathbf{j}_{d-1}, 0) \in H_{d-1} \otimes \{0\} \subset H_d$. For any $\mathbf{j}$ we have $\mathbf{e}_d \in \mathcal{P}(h_\mathbf{j})$ and for $i \in [d-1]$ we have $(-1)^{\mathbf{j}_i} \mathbf{e}_i \in \mathcal{P}(h_\mathbf{j})$.

We first identify a configuration on H_d that allows us to remove any vacancies in the direction $H_d + k\mathbf{e}_d$ for $k \geq 1$ and for which we can apply the exterior condition theorem, Theorem 8.2. Then we use the path method to conclude.

Recall from the construction of the MCEM that we associate a corner of the hypercube to each vacancy type. We call a configuration $\omega \in \Omega$ H_d-good if ω_x for $x \in H_d$ is either in the state of its associated vacancy type or in the neutral state if there is no associated vacancy type, i.e. if $\omega_h = h$ for every $h \in G$ and $\omega_x = \star$ for $x \in H_d \setminus \{G\}$ (see Figure 8.1 for the $d = 3$ case). By the above assumption on G this means that if ω is H_d-good, then any vertex $v \in H_d$ with $v \cdot \mathbf{e}_d = 1$ is in the neutral state, i.e. $\omega_{H_{d-1}\otimes\{1\}} \equiv \star$.

Given an H_d-good ω and a vacancy type $h \in G$ there is a legal path starting from ω and ending in a state η with $\eta_x = h$ for $x \in H_{d-1} \otimes \{1\}$ and $\eta_x = \omega_x$ otherwise. Indeed, assume $h = (0, 0, \dots, 0)$, then we can put h on $\mathbf{e}_d = h + \mathbf{e}_d$ since $\mathbf{e}_d \in \mathcal{P}(h)$. Subsequently, we can put h on any $\mathbf{e}_d + \mathbf{e}_i$ for $i \neq d$ since $\mathcal{P}(h)$ consists of all positive propagation directions. Iterate this procedure adding another $\mathbf{e}_j$ with $j \neq i, d$ and so on until all of $H_{d-1} \otimes \{1\}$ is in state h. By construction this is possible for any $h \in G$.

Then, there is a legal path starting from ω ending in a state η such that $\eta_{H_{d-1}\otimes\{2\}} \equiv \star$. Indeed, this is a consequence of $\mathbf{e}_d$ being a propagation direction of any vacancy type h and the fact that we can bring h to any vertex in $H_{d-1} \otimes \{1\}$ as discussed in the previous paragraph. By reversibility, this implies that we can construct a legal path that puts $H_{d-1} \otimes \{2\}$ into any state.

For any $k \in \mathbb{N}, k \geq 2$ we can iterate this argument to find a legal path from ω to σ where $\sigma_x = \star$ if $x \in \cup_{j\in[2,k]}(H_{d-1} \otimes \{j\})$ and $\sigma_x = \omega_x$ otherwise. By reversibility we can thus find a legal path to *any* σ that agrees with ω outside of $\cup_{j\in[2,k]}(H_{d-1} \otimes \{j\})$.

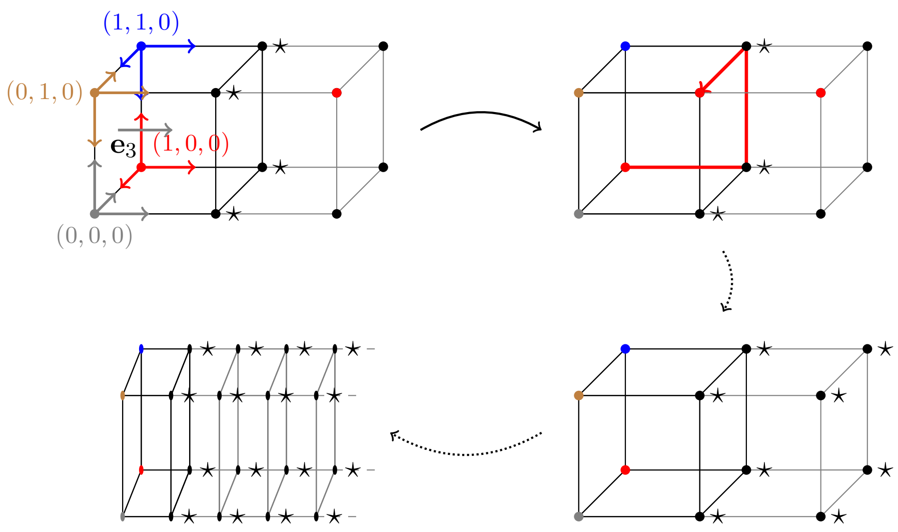

FIGURE 8.1 Path from proof of part (B.i) for $d = 3$. The first image (top left) shows the part on H_d of a H_d-good configuration with the propagation directions of the involved vacancies. To remove the $(0, 0, 1)$-vacancy on $(0, 1, 2)$ we use the red path from the second image. Iterating this procedure to put $\star$ on all the black vertices (of initially arbitrary state) at $(\cdot, \cdot, 2)$ (third image). This procedure iterates to any $(\cdot, \cdot, k)$ for $k \geq 2$ (fourth picture).

We define $(H_d + x)$-good ω analogously to H_d-good ω by translating the conditions to the hypercube translated by $x \in \mathbb{Z}^d$. Let $V_n = \{x \in \mathbb{Z}^d : x \cdot \mathbf{e}_d \geq -n\}$ for $n \in \mathbb{Z}$ so that $\{V_n\}_{n\in\mathbb{Z}}$ is an increasing and exhausting family of subsets of $\mathbb{Z}^d$. With $\mathcal{A}_{x,j} := \{\omega : \omega \text{ is } (H_d + x - (j + 1)\mathbf{e}_d)\text{-good}\}$ we find that the family $\mathcal{A}'_x(N) = \cup_{j\in[N]}\mathcal{A}_{x,j}$ for $N \geq 1$ satisfies the exterior condition with respect to $\{V_n\}_{n\in\mathbb{Z}}$. The support of $\mathcal{A}'_x(N)$ increases linearly in N but the equilibrium failure probability decreases exponentially in N. Thus, we can choose N large enough for Equation (8.1) to hold and with Lemma 8.5 we get

$$\begin{aligned}\mathrm{Var}(f) &\leq 4\sum_{x\in\mathbb{Z}^d}\mu\Big(\mathbb{1}_{\mathcal{A}'_x(N)}\mathrm{Var}_x(f)\Big)\\ &\leq 4\sum_{x\in\mathbb{Z}^d}\sum_{j\in[N]}\mu\Big(\mathbb{1}_{\mathcal{A}_{x,j}}\mathrm{Var}_x(f)\Big)\\ &\leq C(\mathbf{q})\sum_{x\in\mathbb{Z}^d}\sum_{j\in[N]}\sum_{h\in G}\sum_{\omega\in\mathcal{A}_{x,j}}\mu(\omega)(\nabla_x^{(h)}f(\omega))^2.\end{aligned}$$

Fix some $x \in \mathbb{Z}^d$, $j \in [N]$, $h \in G$ and $\omega \in \mathcal{A}_{x,j}$ and assume w.l.o.g. that $\omega_0 = h$. By the above observations and translation invariance of the dynamics we find a legal path $(\sigma^{(1)},\dots,\sigma^{(m)})$ with $m = O(N^2 2^{2d})$, $\sigma^{(1)} = \omega$ and $\sigma^{(m)} = \sigma$ where σ is the state given by $\sigma_x = \star$ and $\sigma_{\mathbb{Z}^d\setminus\{x\}} = \omega_{\mathbb{Z}^d\setminus\{x\}}$. Using the path method gives

$$\mu(\omega)(\nabla_x^{(h)}f(\omega))^2 \leq C(\mathbf{q},m)\mu_{\mathbb{Z}^d\setminus\Lambda_j(x)}(\omega)\mathcal{D}_{\Lambda_j(x)}(f)(\omega)$$

where $\Lambda_j(x)$ is the smallest box containing both the support of $\mathcal{A}_{x,j}$ and the origin. Using that $\Lambda_j(x)$ is finite for any x and j we get

$$\begin{aligned}\mathrm{Var}(f) &\leq C(\mathbf{q},m)\sum_{x\in\mathbb{Z}^d}\sum_{j\in[N]}\sum_{h\in G}\sum_{\omega\in\mathcal{A}_{x,j}}\mu_{\mathbb{Z}^d\setminus\Lambda_j(x)}(\omega)\mathcal{D}_{\Lambda_j(x)}(f)(\omega)\\ &\leq C(\mathbf{q},m)\sum_{x\in\mathbb{Z}^d}\sum_{j\in[N]}\mu(-f\mathcal{L}_{\Lambda_j(x)}f)\\ &\leq C(\mathbf{q},m)\mathcal{D}(f).\end{aligned}$$

By the variational characterisation of the spectral gap we thus have

$$\gamma(G,\mathbf{q}) > 1/C(\mathbf{q},m).$$

which is the claim. □

8.3 G as a star graph: Proof of Theorem 5(B.ii)

By Lemma 8.4, assume w.l.o.g. that $G = \{h_c, h_1, \dots, h_d\}$ where $h_c = 0$ is the central vertex of G and $h_i = \mathbf{e}_i$, $i \in [d]$. We have $\mathcal{P}(h_c) = \{\mathbf{e}_1,\dots,\mathbf{e}_d\}$ and $\mathcal{P}(h_i) = \{\mathbf{e}_1,\dots,\mathbf{e}_{i-1},-\mathbf{e}_i,\mathbf{e}_{i+1},\dots,\mathbf{e}_d\}$ so that the direction $-\mathbf{e}_i$ is unique to h_i.

Let Λ be the equilateral box of side length 2 and origin at 0. We call the vertex $x \in \Lambda$ a corner of Λ if $x_i \in \{0,2\}$ for all $i \in [d]$ and write $F_i = \{x \in \Lambda\colon x_i = 0\}$. For a configuration $\omega \in \Omega$ we say that Λ is good if $\omega_{2h} = h$ for every $h \in G$ and $\omega_x = \star$ for $x \in \cup_i F_i \setminus \{2h\colon h \in G\}$. Analogously define good boxes $\Lambda + x$ for any $x \in \mathbb{Z}^d$.

Lemma 8.8. *If $\omega \in \Omega$ is such that Λ is good and for each $i \in [d]$ there is a smallest $k_i \geq 2$ with $\omega_{\mathbf{v}+k_i\mathbf{e}_i} = h_i$, where we write $\mathbf{v} = \sum_{i\in[d]}\mathbf{e}_i$, then there is a legal path starting at ω and ending a σ such that*

(i) $\Lambda + \mathbf{v}$ is good in σ, and

(ii) $\sigma_x = \star$ *for any* x *between* Λ *and* $\mathbf{v} + k_i\mathbf{e}_i$*, i.e. any* $x \in \cup_i\{\mathbf{v} + j\mathbf{e}_i \colon 3 \leq j \leq k_i - 1\}$ *(if* $k_i \geq 4$*), and*

(iii) ω *and* σ *agree otherwise.*

Proof. We start by showing that there is a legal path that puts any state on $\Lambda \setminus \cup_i F_i$ and then we show how to use this to get that $\Lambda + \mathbf{v}$ is good. The steps are outlined in Figure 8.2 for $d = 2$.

Relax $\Lambda \setminus \cup_i F_i$: Fix an ω as in the claim. Consider the vertex $h_c + \mathbf{v} = \mathbf{v} \in \Lambda$. For any $i \in [d]$ we have $\mathbf{v} - \mathbf{e}_i \in F_i \setminus \{2h \colon h \in G\}$ and thus $\omega_{\mathbf{v}-\mathbf{e}_i} = \star$. Now let $j \in [d]$, $j \neq i$. Since $-\mathbf{e}_j$ is the only negative unit vector in $\mathcal{P}(h_j)$, there is a path[1] from $\mathbf{e}_j$ to $\mathbf{v} - \mathbf{e}_i$ contained in $F_i \setminus \{2h \colon h \in G\}$ consisting only of steps in $\mathcal{P}(h_j)$. Similar considerations apply to paths from the origin containing only steps in $\mathcal{P}(h_c)$. Thus, recalling that $\mathbf{e}_i \in \mathcal{P}(h_j)$, if Λ is good there is a legal path that removes any non-h_i-vacancy from $\mathbf{v}$. Since i was arbitrary any vacancy type on $\mathbf{v}$ can be removed and by reversibility also any vacancy type can be put. Hence, we can also remove any non h_i-vacancy from $\mathbf{v} + \mathbf{e}_i$.

Now use that $\mathbf{v} + \mathbf{e}_i - \mathbf{e}_j$ for $j \neq i$ is in F_j ($F_j \setminus \{2h \colon h \in G\}$ if $d > 2$) again and there is a path contained in $F_j \setminus \{2h \colon h \in G\}$ from $\mathbf{e}_i$ to $\mathbf{v} + \mathbf{e}_i - \mathbf{e}_j$ consisting only of steps in $\mathcal{P}(h_i)$ so that there is a legal path that removes any vacancy from $\mathbf{v} + \mathbf{e}_i$. Since i was arbitrary again we find a legal path that can put or remove any vacancy from $\mathbf{v} + \mathbf{e}$, $\mathbf{e} \in \mathcal{B}$. Analogously, it follows by induction in n that we can remove or put any vacancy type on $x \in \Lambda \setminus \cup_i F_i$ with $\|x - \mathbf{v}\|_1 = n$, where we just proved the base case $n = 1$.

Make $\Lambda + \mathbf{v}$ good: For $i \in [d]$ we want to find a legal path which puts h_i on $2\mathbf{e}_i + \mathbf{v}$ and, if $k_i \geq 4$, also puts the neutral state $\star$ on $\{\mathbf{v} + j\mathbf{e}_i \colon 3 \leq j \leq k_i - 1\}$. Then remove any other vacancies from $\cup_i(F_i + \mathbf{v}) \setminus \{2h + \mathbf{v} \colon h \in G\}$.

Let $i \in [d]$ and assume w.l.o.g. that $k_i > 2$ since otherwise the h_i-vacancy is already at the correct position for $\Lambda + \mathbf{v}$ to be good. We already know that we can put any vacancy on $\mathbf{v}$ and since $\mathbf{e}_i \in \mathcal{P}(h)$ for every $h \in G \setminus \{h_i\}$ there is a legal path that removes any vacancy from $\{\mathbf{v} + j\mathbf{e}_i \colon j \in [1, k_i - 1]\}$ (that are by assumption not h_i-vacancies since k_i is the smallest integer such that $\omega_{\mathbf{v}+k_i\mathbf{e}_i} = h_i$). Then, use that $-\mathbf{e}_i \in \mathcal{P}(h_i)$ to bring the h_i from $\mathbf{v} + k_i\mathbf{e}_i$ to $\mathbf{v} + 2\mathbf{e}_i$ and put $\star$ in between $\mathbf{v} + 2\mathbf{e}_i$ and $\mathbf{v} + k_i\mathbf{e}_i$.

Since i was arbitrary we can put h_i on $\mathbf{v} + 2\mathbf{e}_i$ for any i. Using again that we can put $\mathbf{v}$ into any state we can, in particular, put h_c on $\mathbf{v}$. Thus, we get a legal path that puts h on $\mathbf{v} + 2h$ for each $h \in G$ and $\star$ on $\cup_i\{\mathbf{v} + j \colon j \in [3, k_i - 1]\}$ where the final state still agrees with ω outside these vertices.

Let $x \in (F_i + \mathbf{v}) \setminus \Lambda$ such that x is not a translated corner $\mathbf{v} + 2\mathbf{e}_j$, $j \in [d]$. Then $x_i = 1$, further there is at least a $j_1 \in [d]$ with $x_{j_1} = 3$, at least a $j_2 \neq j_1$ with $x_{j_2} \geq 2$ and $x_j \in \{1, 2, 3\}$ otherwise.

Assume that there is exactly one such j_1 and j_2 with $j_2 = 2$. Then $x - \mathbf{e}_{j_1}$ is in $\Lambda \setminus \cup_i F_i$ and by the above observations we thus find a legal path that removes any h-vacancy from x for $h \in G \setminus \{h_{j_1}\}$. To remove an h_{j_1}-vacancy use that $x - \mathbf{e}_{j_2} = 2\mathbf{e}_{j_1} + \mathbf{v}$ on which we already know that there is a legal path to put an h_{j_1}-vacancy.

Building on this, the argument is analogous if $j_2 = 3$ or there are two j with $x_j = 2$. The claim follows by iterating these analogous arguments. □

The Lemma tells us that we can move a good Λ in the direction $\mathbf{v}$, given enough non-central vacancies outside of Λ. To satisfy the exterior condition this is too lose a condition as we cannot always assume that we find these vacancies for each step. The next Lemma gives another construction that does not require new vacancies after every step.

[1] We use the term *path* to mean paths on the lattice $\mathbb{Z}^d$ or on subsets Λ and the term *legal path* to mean paths of configurations in Ω that are legal in the G-MCEM.

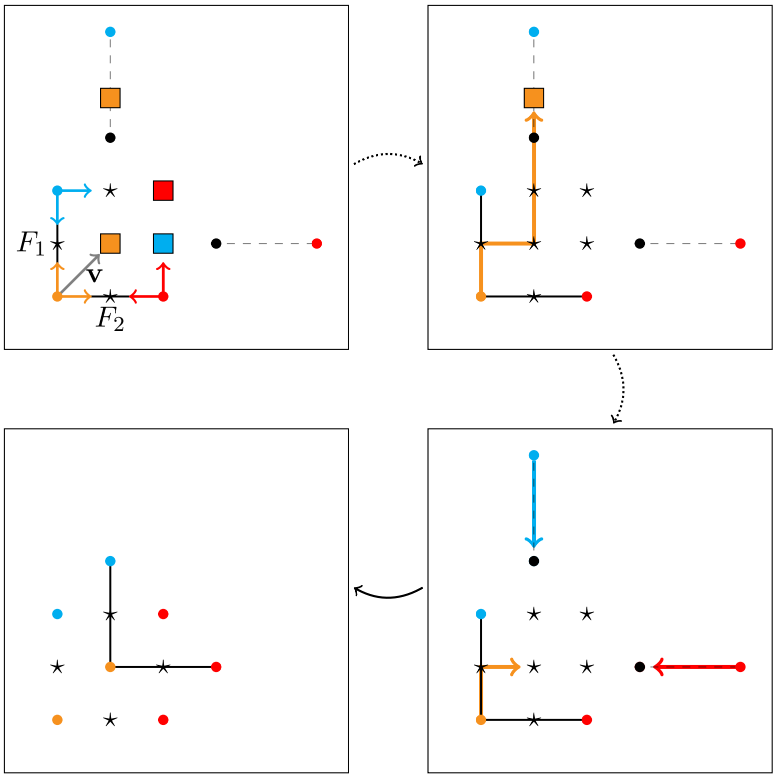

FIGURE 8.2 Example configuration from Lemma 8.8 in the top-left. The black circles indicate the vertices to which we need to bring the $(0,1)$-vacancy (blue) resp. $(1,0)$-vacancy (red). The squares indicate the vacancies we replace with $\star$ in the course of the proof. From the first to the second image any vacancy on $\Lambda \setminus \cup_i F_i$ is replaced with $\star$. For the third image we then bring the necessary vacancies to $\Lambda + \mathbf{v}$ and the final image shows how this gives a good configuration on $\Lambda + \mathbf{v}$.

Lemma 8.9. *Fix an $N \in 2\mathbb{N}$, $N \geq 4$. Let $\omega \in \Omega$ be such that Λ is good and for each $i \in [d]$ there is a $k_i \in [N, 3N/2]$ with $\omega_{\mathbf{v}+k_i\mathbf{e}_i} = h_i$ and such that $\omega_y = \star$ for each $y \in \{\mathbf{v} + n\mathbf{e}_i : n \in [k_i - 1]\}$. Then, there is a legal path starting at ω and ending at σ such that $\Lambda + (N-2)\mathbf{v}$ is good and that agrees with ω otherwise.*

Proof. Figure 8.3 illustrates a state ω as in the claim and the steps of the following proof. We start by clearing the line $\{2\mathbf{v} + n\mathbf{e}_i : n \in [N-1]\}$ of any vacancies and then move the good box by $\mathbf{v}$ so that we recover the initial situation and can iterate the argument.

Fix an $i \in [d]$ and $j \neq i$. We can bring the h_i-vacancy from $\mathbf{v} + k_i\mathbf{e}_i$ to $\mathbf{v} + n\mathbf{e}_i$ for any $n \in [N]$. Thus, we can remove any h_i-vacancy from $\{\mathbf{v} + n\mathbf{e}_i + \mathbf{e}_j : n \in [N]\}$. Since $\mathbf{v} + \mathbf{e}_j \in \Lambda \setminus \cup_i F_i$, we can put any h-vacancy on it. Thus, there is a legal path to remove any vacancy from $\{\mathbf{v} + n\mathbf{e}_i + \mathbf{e}_j : n \in [N]\}$ using that $\mathbf{e}_i \in \mathcal{P}(h)$ for $h \neq h_i$. The chosen $j \neq i$ was arbitrary so we can remove any vacancy from any such line.

Prove analogously that we can remove any vacancy from $\{\mathbf{v} + n\mathbf{e}_i + \sum_{i \in I} \mathbf{e}_i : n \in [N-1], I \subset [d] \setminus [i], |I| = m\}$ for any $m \leq d-1$ by induction in m. This is done by using that if the statement holds for $m-1$, then we can remove any h_i vacancy from the line with some $|I| = m$ by subtracting $\mathbf{e}_j$ for $j \in I$ so that we land on a line with $|I| = m-1$ and finally use that $\mathbf{v} + \sum_{i \in I} \mathbf{e}_i \in \Lambda \setminus \cup_i F_i$ again,

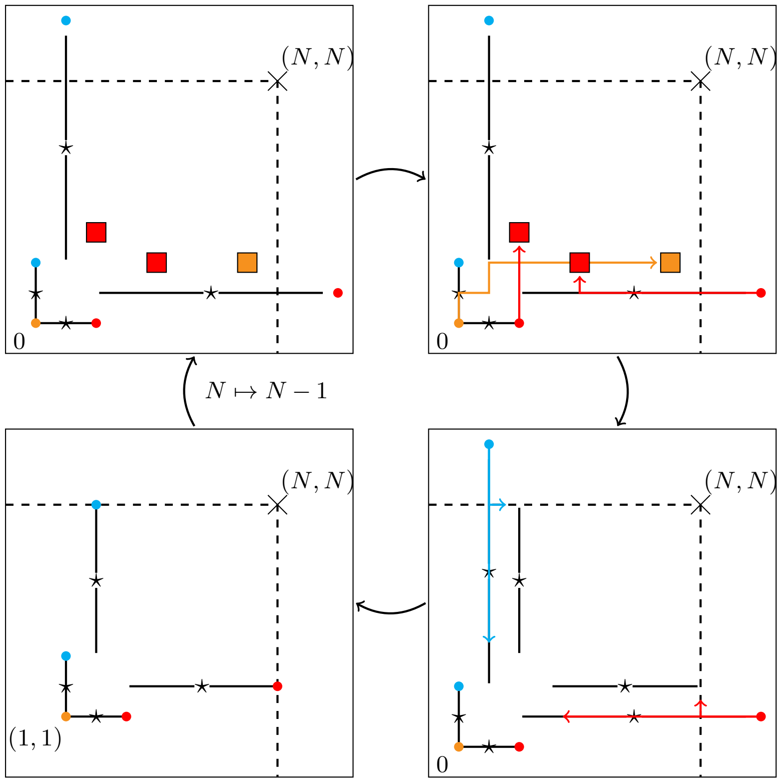

FIGURE 8.3 The first image (top left) shows an example ω as in Lemma 8.9 for two dimensions using the colour and shape code from Figure 8.1. The second image shows the paths used to get rid of any vacancies on $\{2\mathbf{v}+j\mathbf{e}_i : j \in [n]\}$. The third image then shows the paths to move the good $\cup_i F_i$ and put the $(0,1)$- resp. $(1,0)$-vacancy at the end of $\{2\mathbf{v}+j\mathbf{e}_i : j \in [n]\}$. The fourth image then shows how the resulting state is the same as in Lemma 8.9 translated by $\mathbf{v}$ so we can iterate the proof by setting $N \mapsto N-1$.

resulting in a legal path that removes any vacancy.

In particular, we can remove any vacancy from $y \in 2\mathbf{v}+(n-1)\mathbf{e}_i$ for $n \in [N]$ since $y = \mathbf{v}+n\mathbf{e}_i+\sum_{i\in[d]\setminus\{\mathbf{e}_i\}} \mathbf{e}_i$ is included above for $m = d-1$. The choice of $i \in [d]$ was arbitrary so that we can construct a legal path that ends in a state $\sigma^{(1)}$ on which $\cup_i\{2\mathbf{v}+n\mathbf{e}_i : n \in [N-2]\}$ is in the neutral state and $2\mathbf{v}+(N-1)\mathbf{e}_i$ has an h_i-vacancy.

Further, by Lemma 8.8 there is a path starting at $\sigma^{(1)}$ and ending in a state $\sigma^{(2)}$ in which $\Lambda+\mathbf{v}$ is good and which does not change $\cup_i\{2\mathbf{v}+n\mathbf{e}_i : n \in [N-1]\}$. The state $\sigma^{(2)}$ is now in the configuration of the claim for $N-1$ so that we can iterate the proof until we find a legal path that ends in a state with $\Lambda+(N-2)\mathbf{v}$ good. □

With this we can come to the proof of the theorem.

Proof of (B.ii). We start by defining an event with which we can apply the exterior condition theorem, Theorem 8.2, and that allows us to use Lemma 8.8 and Lemma 8.9. For $N \in 4\mathbb{N}$ define the event $\mathcal{E}^{(N)}$ as the set of configurations ω such that

- For any $i \in [d]$ there is an $n_i \in [N, 3N/2]$ such that $\omega_{-N\mathbf{v}+n_i\mathbf{e}_i} = h_i$.

- For each $y \in \cup_i\{-N\mathbf{v} + j\mathbf{e}_i : j \in \{0, \dots, n_i - 1\}\}$ there is an $m_y \geq 2$ such that $\Lambda + y - m_y\mathbf{v}$ is good and such that for each $j \in [m_y - 1]$ and each $i \in [d]$ there is an h_i-vacancy on $\{y - j\mathbf{v} + k\mathbf{e}_i : k \in [N/4]\}$.

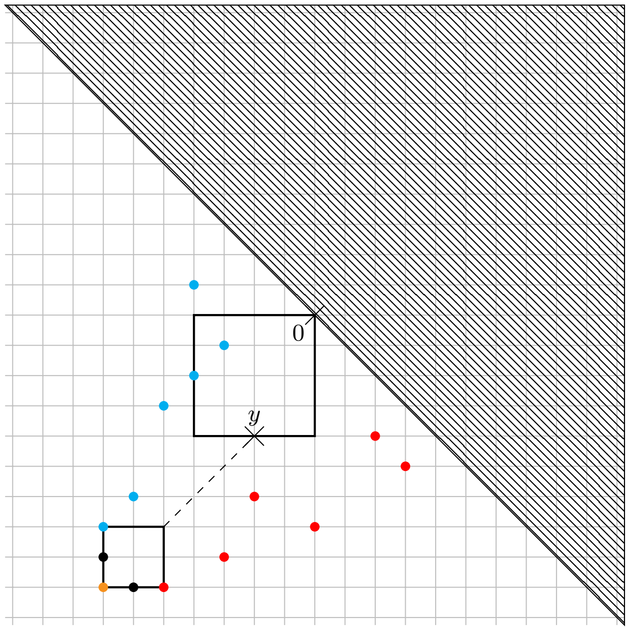

FIGURE 8.4 Example configuration in the event $\mathcal{E}^{(N)}$ in $d = 2$ and for $N = 4$. The dashed region is V_0, the bigger square is the square on which we use Lemma 8.9. The smaller square is the good box needed with $m_y = 5$ for the representative y we chose on the boundary of the big square. Contrary to the definition of $\mathcal{E}^{(N)}$, the h_i vacancies for the good boxes have a distance further than $N/4$ from $\{y - j\mathbf{v} : j \in [5]\}$ to illustrate the example better since $N/4 = 1$ results in a diagonal of vacancies.

See Figure 8.4 for an illustration in $d = 2$. Let $\mathcal{E}_x^{(N)}$ be the correspondingly translated event for $x \in \mathbb{Z}^d$. We start by checking that the family $\{\mathcal{E}^{(N)}\}_{N\in\mathbb{N}}$ satisfies the exterior condition, then calculate the failure probability and conclude with the path method.

Exterior condition On $\mathbb{Z}^d$ consider the $d-1$ dimensional hyperplane U_0 perpendicular to $\mathbf{v}$ that goes through the origin. We claim that $\mathrm{Supp}(\mathcal{E}^{(N)}) \cap U_0 = \emptyset$. Indeed, any vertex $x \in U_0$ is characterized by the equation $x \cdot \mathbf{v} = 0$. To have $-N\mathbf{v} + \ell\mathbf{e}_i \in U_0$ we thus need $\ell = N \cdot d$. By definition of $\mathcal{E}^{(N)}$ the furthest vertices on which we look for vacancies are on $y + k \leq 7N/4$ so that $\mathrm{Supp}(\mathcal{E}^{(N)}) \cap U_0 = \emptyset$ for $d \geq 2$ and since the support is connected we also have $\mathrm{Supp}(\mathcal{E}^{(N)}) \subset V_0^c$ where $V_0 := \cup_{\ell=1}^{\infty}(U_0 + \ell\mathbf{v})$. Then, the family $\{\mathcal{E}_x^{(N)}\}_{x\in\mathbb{Z}^d}$ satisfies the exterior condition w.r.t. $\{V_n\}_{n\in\mathbb{Z}}$ where $V_n = V_0 - n\mathbf{v}$.

Failure probability To apply the exterior condition theorem it remains to show that Equation (8.1) holds. The support of $\mathcal{E}_x^{(N)}$ is $O(N)$ while the failure probability is, by translation invariance, upper bounded by the union bound of $O(N)$ events that require $O(N)$ Bernoulli-trials of probability $q_{\min}$ to fail, so that we find constants $C, \kappa > 0$ giving

$$\mathrm{Supp}(\mathcal{E}^{(N)})\mu(1 - \mathbb{1}_{\mathcal{E}^{(N)}}) \leq CN^2(1 - q_{\min})^{\kappa N}.$$

Now choose N (depending on $\mathbf{q}$, C and κ) big enough so that Equation (8.1) holds. Theorem 8.2 then gives

$$\mathrm{Var}(f) \leq 4 \sum_{x\in\mathbb{Z}^d} \mu(\mathbb{1}_{\mathcal{E}_x^{(N)}} \mathrm{Var}_x(f)).$$

Path method The proof concludes analogously to the proof of (B.i) once we have defined the paths that allow us to remove or put any vacancy type on x. W.l.o.g. consider only the case $x = 0$ and fix $\omega \in \mathcal{E}^{(N)}$ with the associated $\{n_i\}_i$ and $\{m_y\}_y$ from the definition of $\mathcal{E}^{(N)}$. For each i we want to remove any vacancy from $y \in \cup_i\{-N\mathbf{v} + j\mathbf{e}_i : j \in \{1, \dots, n_i - 1\}\}$, and we start with the biggest one in the $\prec$ order, i.e. $y = -N\mathbf{v} + (n_i - 1)\mathbf{e}_i$.

We have that $\Lambda + y - m_y\mathbf{v}$ is good and that for each $i \in [d]$ there is an h_i-vacancy on $\{y - (m_y - 1)\mathbf{v} + k\mathbf{e}_i : k \in [N/4]\}$. Thus, by Lemma 8.8 there is a legal path that only depends on and changes the states of the vertices on $\Lambda + y - m_y\mathbf{v}$ and $\{y - (m_y - 1)\mathbf{v} + k\mathbf{e}_i : k \in [N/4]\}$ and which ends in a state in which $\Lambda + y - (m_y - 1)\mathbf{v}$ is good. Since the vertices on $\{y - (m_y - 2)\mathbf{v} + k\mathbf{e}_i : k \in [N/4]\}$ remain the same, by definition of $\mathcal{E}^{(N)}$ we have again h_i-vacancies on them.

We can iteratively apply Lemma 8.8 resulting in a legal path that ends in a state where $\Lambda + y - 2\mathbf{v}$ is good, so that y is a corner of this box. Then, we can flip y to any state and in particular to the neutral state $\star$.

By reversibility and using that all the flips on this path are independent of the state of y we can reverse the transitions to get a path from ω to σ where $\sigma_y = \star$ and $\sigma_x = \omega_x$ for $x \neq y$.

Let $y' = -N\mathbf{v} + (n_i - 2)\mathbf{e}_i$ and note that $y \notin \cup_{m\in[2,m_{y'}]} \cup_{k\in[N/4]} \{y' - m\mathbf{v} + k\mathbf{e}_i\}$ (which in $d = 2$ only works since we go from largest to smallest), so that the relevant surrounding configuration guaranteed by $\mathcal{E}^{(N)}$ is untouched by the first step and repeating the same construction for y' leaves y untouched.

Thus, we can repeat the same path construction, using $\omega \in \mathcal{E}^{(N)}$, to remove the vacancy of y' and iterate to get a legal path that ends in a state where $\omega_y = \star$ for each $y \in \cup_i\{-N\mathbf{v} + j\mathbf{e}_i : \{0, \dots, n_i - 1\}\}$ and finally add a piece to the path that makes $\Lambda - (N + 1)\mathbf{v}$ good by the same argument just instead of putting the neutral state and moving the good box back by reversibility we keep the good box at $\Lambda - (N + 1)\mathbf{v}$. From this state we can use Lemma 8.9 to find the desired legal path with which we can conclude analogously to (B.i). □

Chapter 9

Spectral gap bounds for the two-dimensional MCEM: Proof of Theorem 6

The upper bound in Theorem 6 follows by Lemma 8.4. The steps to prove the corresponding lower bound are analogous to the proof of Theorem 5(B). The main difference is that we have to be careful about the cost of our intermediate steps. Where before we were fine estimating $\gamma(G, \mathbf{q}) > C(\mathbf{q})$ for some constant we now want a specific bound. This means that we need some intricate constructions and events.

In Section 9.1 we construct a grid of points together with paths connecting them so that we get a set isomorphic to a box in $\mathbb{Z}^2$ on which the vacancies can travel such that the two-dimensional motion on the grid of points dominates the one-dimensional motion between the points. This is enough to prove part (3.i) of Theorem 6 in Section 9.2 in which, we recall, there is no frequent vacancy type. To prove (3.ii) in Section 9.3, for which there is one frequent vacancy type, we do the same construction but this time on the renormalised lattice of boxes and identify box configurations that travel like the infrequent vacancy types.

Finally, in the proof of Theorem 6(3.iii), contained in Section 9.4, where there are two frequent vacancy types we need a different construction altogether that exploits the fact that the frequent vacancy types have a common direction and are disseminated throughout the lattice to identify high probability configurations that allow the infrequent vacancy type to move two-dimensionally.

9.1 Preliminary constructions

Note that by Lemma 8.4 the cases (3.i) and (3.ii) imply the cases (2.i) and (2.ii). Using this and symmetry considerations, w.l.o.g. we can assume in the following that $G = \{(1,1), (0,0), (0,1)\}$. We call the associated MCEM the ABC-model and call $A = (0,0)$, $B = (1,1)$, $C = (0,1)$ and $D = (1,0)$. As noted in the introduction, by Lemma 8.4 we have

$$\lim_{q_{\min} \to 0} \frac{\gamma(G, \mathbf{q})}{\gamma_2(q_{\min})} \leq 1$$

to prove Theorem 6 we thus need the corresponding lower bound.

Analogously to bounding the spectral gap from zero our strategy for finding good lower bounds on the spectral gap relies on the exterior condition theorem, Theorem 8.2. Fix a $G \subset H_2$ and consider a family $\{\mathcal{A}_x\}_{x \in \mathbb{Z}^2}$ of events that satisfies the requirements of the exterior condition theorem so that with

Lemma 8.5 we have

$$\mathrm{Var}(f) \leq 4 \sum_{x\in\mathbb{Z}^2} \mu(\mathbb{1}_{\mathcal{A}_x}\mathrm{Var}_x(f)) \leq \frac{4}{p}\sum_{h\in G}\sum_{x\in\mathbb{Z}^2}\mu\Big[\mathbb{1}_{\mathcal{A}_x} pq_h(\nabla_x^{(h)} f)^2\Big].$$

Since $pq_h(\nabla_x^{(h)} f)^2 = \mathrm{Var}_x(f\mathbb{1}_{\{\star,h\}}) =: \mathrm{Var}_x(f \mid \{\star,h\})$ we can treat the transition for each vacancy type separately. The main difficulty in finding good lower bounds on the spectral gap is then to identify events $\mathcal{A}_x$ that satisfy the exterior condition, have a low failing probability and such that for each $h \in G$ we have

$$\mu[\mathbb{1}_{\mathcal{A}_x}\mathrm{Var}_x(f \mid \{\star, h\})] \leq 2^{\theta^2_{q_{\min}}(1+\varepsilon)/4}\mu[\mathcal{D}_{\Lambda_h}(f)], \tag{9.1}$$

for some Λ_h such that the overlap of the various Λ_h for the different x (and thus the overcounting term) can be absorbed into the ε in $2^{\theta^2_{q_{\min}}(1+\varepsilon)/4}$ for $q_{\min}$ small enough. We do this by defining events $\mathcal{A}_x^{(h)}$ for each $h \in G$ and setting $\mathcal{A}_x = \cap_{h\in G}\mathcal{A}_x^{(h)}$. Each $\mathcal{A}_x^{(h)}$ is defined such that it allows the rewriting of the local variance with indicator $\mathbb{1}_{\mathcal{A}_x^{(h)}}$ to a Dirichlet form by using a mixture of auxiliary models that behave like the standard one- or two-dimensional East model and the path method.

This section introduces the construction of the grid that we use for the proofs of part (3.i) and (3.ii). In Section 9.1.1 we do the geometric construction and in Section 9.1.2 we introduce the events together with their failing probability for which we use the exterior condition theorem.

9.1.1 Geometric construction

Let us start by introducing the notion of oriented paths for $h \in G$.

Definition 9.1 (h-paths)**.** For $h \in H_2$ we say that $\Gamma = (x_1, \dots, x_n) \subset \mathbb{Z}^2$ is an h-path if $x_i - x_{i+1} \in \mathcal{P}(h)$ for $i \in [n-1]$, i.e. starting from x_n we can reach x_1 staying on Γ and only using steps in $\mathcal{P}(h)$.

Remark 9.2. Note that we want $x_i - x_{i+1}$ to be a propagation direction of $\mathcal{P}(h)$ instead of the more intuitive direction from x_i to x_{i+1} (i.e. $x_{i+1} - x_i$). Defining it this way we can find an h-path starting from some vertex $x \in \mathbb{Z}^d$ and ending in a vertex containing an h-vacancy which can then travel on the h-path back to x.

We build the h-grid first for B-vacancies and then explain how to generalise to $h \in \{A, C\}$. We do the construction incrementally by starting with a base cell for B-vacancies.

Definition 9.3 (B-Base cell Q)**.** Let $\ell \in 8\mathbb{N}$. Define $D^{(1)} \subset \mathbb{Z}^2$ as the B-path starting at $\mathbf{e}_1 + 3\mathbf{e}_2$ that first does an $\mathbf{e}_1$-step, then zigzags north and east for 2 steps respectively until ℓ steps east have been made with the last step being a single one. Then define $D^{(2)} \subset \mathbb{Z}^2$ as the path starting again at $\mathbf{e}_1 + 3\mathbf{e}_2$ which starts with 4 steps north, goes one step east and then zigzags 8 steps north and one step east until ℓ steps north have been made with the last step 4 long instead of 8. Then, define $D^{(3)} = D^{(1)} + \ell/8\mathbf{e}_1 + \ell\mathbf{e}_2$ and $D^{(4)} = D^{(2)} + \ell\mathbf{e}_1 + \ell\mathbf{e}_2$, i.e. the paths $D^{(1)}$ resp. $D^{(2)}$ shifted to start at the end point of $D^{(2)}$ resp. $D^{(1)}$. We then define the *B-base cell Q with side length ℓ* as the set of vertices enclosed by and including the boundaries $D^{(i)}$ for $i \in [4]$. We refer to $D^{(i)}$ as the bottom, left, top and right boundary of Q for $i = 1, 2, 3, 4$ respectively (see left side of Figure 9.1).

For the rest of this section fix a side length ℓ. In this base cell we define the notion of interior crossing paths in the horizontal and vertical direction.

Definition 9.4 (Interior B-crossings and cross)**.** Let Q be the B-base cell. We say that a B-path $(x^{(1)}, \dots, x^{(n)}) \subset Q$ is a *vertical interior B-crossing for Q* if $x^{(1)} \in D^{(1)}$, $x^{(n)} \in D^{(3)}$ and $x^{(i)} \notin$

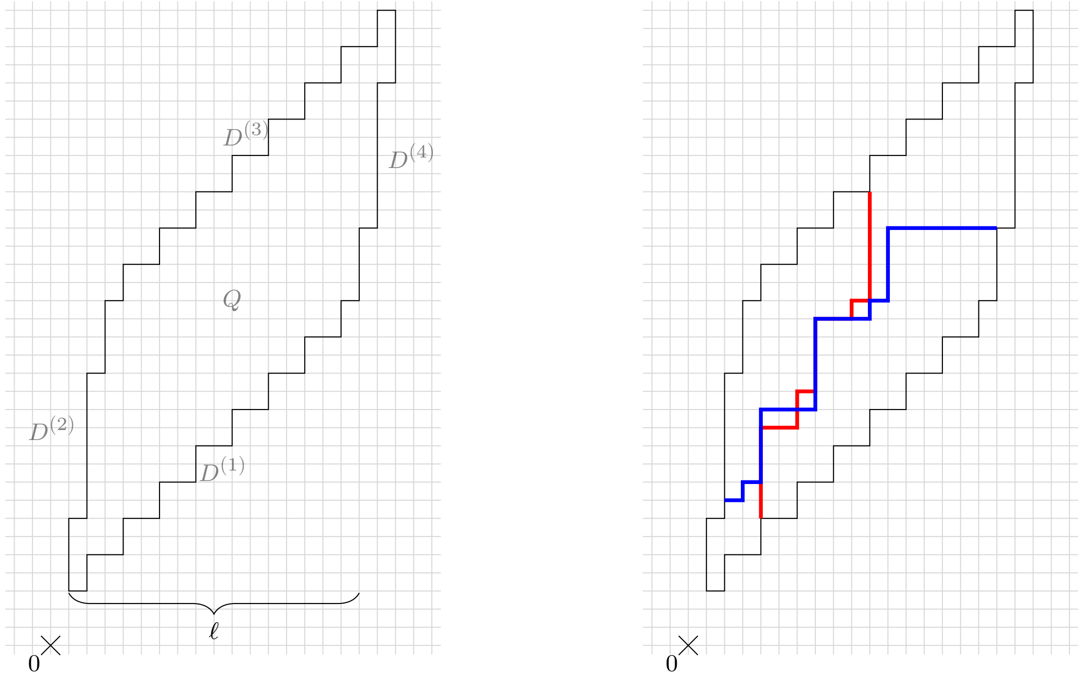

FIGURE 9.1 Left and right: B-Base cell Q with side length $\ell = 16$. Left: Notation as introduced in Definition 9.3. Right: Base cell Q with cross as in Definition 9.4 with horizontal interior crossing in blue and vertical interior crossing in red.

$\bigcup_{i\in[4]} D^{(i)}$ for $i \in [2, n-1]$. Similarly, we say that it is a *horizontal interior B-crossing* if $x^{(1)} \in D^{(2)}$, $x^{(n)} \in D^{(4)}$ and $x^{(i)} \notin \bigcup_{i\in[4]} D^{(i)}$ for $i \in [2, n-1]$ (see right side of Figure 9.1). We call a pair $\mathcal{C}_0 = (\mathcal{C}_0^{(v)}, \mathcal{C}_0^{(h)})$ of a vertical interior crossing and horizontal interior crossing of Q a *cross in Q*.

We translate the cell Q to construct larger square grids of cells.

Definition 9.5 ($Q_{i,j}$)**.** Let $\mathbf{b}_1 = \ell(\mathbf{e}_1 + \mathbf{e}_2)$ and $\mathbf{b}_2 = (\ell/8)\mathbf{e}_1 + \ell\mathbf{e}_2$. For $i, j \in \mathbb{Z}$ we then let $Q_{i,j} = Q_0 + i\mathbf{b}_1 + j\mathbf{b}_2$. Given a *square side length* $N \in \mathbb{N}$ we define the *rectangle of grids* $\mathcal{Q}^{(B)}$ as

$$\mathcal{Q}^{(B)} = \bigcup_{(i,j)\in[0,N]^2} Q_{i,j} \ .$$

Remark 9.6. Notice that $Q_{0,0} = Q$ and that neighbouring cells share a boundary.

In what follows consider the square side length $N \in \mathbb{N}$ fixed. On sets of neighbouring cells we introduce a notion of hard interior B-crossing, as opposed to the local one which only dealt with paths in one cell.

Definition 9.7 (B-strips and hard interior B-crossing)**.** For $i \in [0, N]$ we call the set of cells

$$Q_i^{(v)} = \bigcup_{j\in[0,N]} Q_{i,j}$$

the *i-th vertical B-strip* and for $j \in [0, N]$ we define the *j-th horizontal B-strip* as

$$Q_j^{(h)} = \bigcup_{i\in[0,N]} Q_{i,j} \ .$$

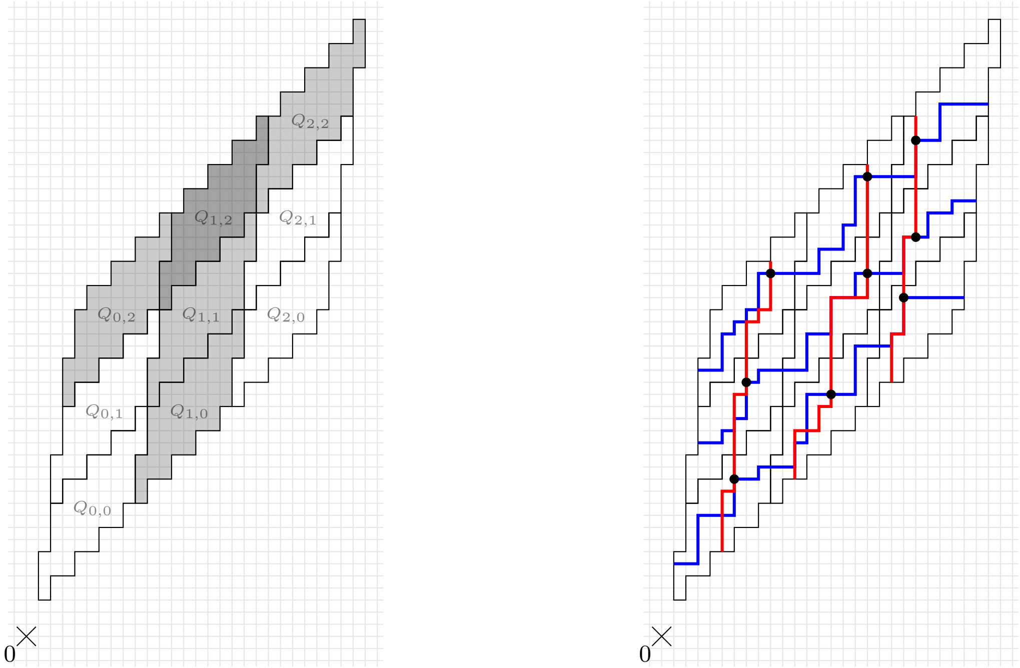

FIGURE 9.2 Left: $Q_{i,j}$ for $i \in [0,2]$ and $j \in [0,2]$, side length $\ell = 8$. The first vertical and second horizontal strip are shaded in gray. Right: A grid $\mathcal{C}$ with $N = 2$. The hard vertical interior crossings are red and hard horizontal interior crossings are blue and the intersections points $X(\mathcal{C})$ are black. Notice that the colours here have nothing to do with the vacancy colours and are just used to distinguish better horizontal from vertical paths.

A B-path $\Gamma \subset Q_i$ is a *hard vertical interior B-crossing of Q_i* if $\Gamma \cap Q_{i,j}$ is a vertical interior B-crossing of $Q_{i,j}$ for any $j \in [0,N]$. Analogously for *hard horizontal interior B-crossings* (see Figure 9.2).

The set of hard interior crossings induce a grid $\mathcal{C}$.

Definition 9.8 (B-grids)**.** For $i \in [0,N]$ let $\mathcal{C}_i^{(v)}$ be a hard vertical interior crossing for the i-th vertical B-strip and for $j \in [0,N]$ let $\mathcal{C}_j^{(h)}$ be a hard horizontal interior crossing of the j-th horizontal strip. We call $\mathcal{C} = (\mathcal{C}^{(v)}, \mathcal{C}^{(h)})$ a *B-grid of $\mathcal{Q}^{(B)}$* where $\mathcal{C}^{(v/h)} = \{\mathcal{C}_i^{(v/h)}\}_{i\in[0,N]}$. Given a grid $\mathcal{C}$ of $\mathcal{Q}^{(B)}$ we call $\mathcal{C}_{i,j} = (\mathcal{C}_i^{(v)}, \mathcal{C}_j^{(h)})$ the cross induced in $Q_{i,j}$.

The intersection points of the induced crosses in each $Q_{i,j}$ form a set that is isomorphic to an equilateral box in $\mathbb{Z}^2$.

Definition 9.9 (Intersection points associated to grid)**.** Given a B-grid $\mathcal{C}$ of $\mathcal{Q}^{(B)}$ we denote by $x_{i,j}$ the highest point in $\mathcal{C}_i^{(v)} \cap \mathcal{C}_j^{(h)}$ in the $\prec^{(B)}$-partial order[1] and call it an intersection point of $\mathcal{C}$. We write $X(\mathcal{C})$ for the set of intersection points. We call $x_{i,j}$ and $x_{i',j'}$ *neighbours in $X(\mathcal{C})$* if (i,j) and (i',j') are neighbours in $[0,N]^2$. Analogously we call $x_{i',j'}$ an *oriented neighbour of $x_{i,j}$ in $X(\mathcal{C})$* if $x_{i',j'}$ and $x_{i,j}$ are neighbours in $X(\mathcal{C})$ such that $(i',j') \prec^{(B)} (i,j)$. We call $x_{i+1,j}$ (if it exists) the *east neighbour of $x_{i,j}$ in $X(\mathcal{C})$* and $x_{i,j+1}$ (if it exists) the *north neighbour of $x_{i,j}$ in $X(\mathcal{C})$* and analogously for the south and west neighbours.

[1] The $\prec^{(B)}$-partial order corresponds to the usual order where $x \prec^{(B)} y$ if $x_i \leq y_i$ for all $i \in [d]$, we write $\prec^{(B)}$ to make it easier to generalise to A- and C-grids.

Remark 9.10. The $\prec^{(B)}$-ordering is only partial but since we look at intersection points of B-paths there is always a unique highest point on $\mathcal{C}_i^{(v)} \cap \mathcal{C}_j^{(h)}$, and since $\mathcal{C}$ induces a cross in each $Q_{i,j}$, $x_{i,j}$ is well defined for any $i, j \in [0, N]$.

The A-base cell is defined analogously by exchanging the role of $\mathbf{e}_1$ with $-\mathbf{e}_2$ and $\mathbf{e}_2$ with $-\mathbf{e}_1$ and for the C-base cell exchange $\mathbf{e}_1$ with $-\mathbf{e}_1$. Do the analogous exchanges in the following definitions for h-crossings and h-grids. When changing the base vectors like this the horizontal A-crossing would cross the base cell vertically so change the names appropriately.

Further, this construction can be translated to be based at any $x \in \mathbb{Z}^2$ by replacing the origin in the definitions with x. We will denote this as an explicit argument so $Q_{i,j}(x) := Q_{i,j} + x$. Since by translation invariance we can apply the results for the origin to any $x \in \mathbb{Z}^2$ this notation is rarely used.

Consider the set V_0 given by the points $x \in \mathbb{Z}^2$ such that $-x_1 + x_2 \leq 0$ (i.e. the set that is 'below' the main diagonal going through the origin) and for $n \in \mathbb{Z}$ let $V_n = V_0 + (-n, n)$, then $\{V_n\}_{n\in\mathbb{Z}}$ is an increasing and exhausting set of $\mathbb{Z}^2$. The following Lemma is the principal reason to construct the h-grid as we did.

Lemma 9.11. *Let $\mathcal{A}_x$ be an event with support in $\cup_{h\in[A,B,C]} \mathcal{Q}_x^{(h)}$, then the family $\{\mathcal{A}\}_{x\in\mathbb{Z}^2}$ satisfies the exterior condition w.r.t. $\{V_n\}_{n\in\mathbb{Z}}$.*

Proof. Follows from the construction of the grids. □

9.1.2 Crossing probabilities and grid relaxation

Let $\mathbf{q}$ be a parameter set for the ABC-model and set $\ell = \lceil \theta_B^{3/2} \rceil$, $N = 2^{\lceil \theta_B/2 + \log_2(\theta_B) \rceil}$ as the parameters for any base cells and grids. The goal for this section is to define an event so that we can use the exterior condition theorem, Theorem 8.2.

We say that a set Λ is *B-traversable* if it does not contain A or C vacancies, and we define correspondingly A- and C-traversability. The event for which we want to apply the exterior condition theorem will require the existence of an appropriately traversable grid $\mathcal{C}$ for each vacancy type so let us upper bound the probability of not finding B-traversable B-crossings as a first step.

Lemma 9.12. *Let $\mathcal{A}$ be the event of finding a B-traversable hard interior B-crossing in a strip Q. If $\mathbf{q}$ is such that $q_A + q_C \to 0$ as $q_B \to 0$ then we find a constant $C > 0$ so that*

$$\mu(\mathcal{A}^c) \leq C 2^{-\theta_B^{3/2}}$$

for q_B small enough.

Proof. We follow the arguments from [43] to apply a Peierls-type argument. We will deal with the vertical case first, the horizontal one being analogous. Consider a vertical strip $Q_i^{(v)}$ with left boundary $D^{(2)}$ and right boundary $D^{(4)}$. Define on it the dual graph Q_i^* as the faces of $Q_i^{(v)}$, i.e. the graph given by

$$Q_i^* = \{x^* \in Q_i^{(v)} + 1/2(\pm\mathbf{e}_1 \pm \mathbf{e}_2) \colon \|\{x \in Q_i^{(v)} \colon \|x^* - x\|_1 = 1\}\| = 4\},$$

with neighbourhood relations induced by $\mathbb{Z}^2 + 1/2(\mathbf{e}_1 + \mathbf{e}_2)$. We define the left boundary $D^{(2,*)}$ as the set of $x^* \in Q_i^*$ for which there exists an $x \in D^{(2)}$ such that $\|x^* - x\|_1 = 1$ and analogously for the right boundary $D^{(4,*)}$ with $D^{(4)}$. Say that the horizontal directed edge $(x^*, x^* + \mathbf{e}_1)$ in Q_i^* is *closed* in a configuration $\omega \in \Omega$ if $x^* + 1/2(\mathbf{e}_1 + \mathbf{e}_2)$ (north-east corner) is not B-traversable, i.e. has an A- or C-vacancy and *open* otherwise. Similarly for the vertical edge $(x^*, x^* + \mathbf{e}_2)$ with vertex $x^* + 1/2(\mathbf{e}_1 - \mathbf{e}_2)$

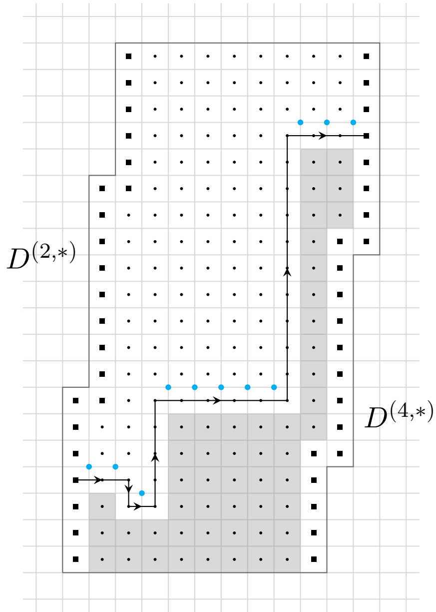

FIGURE 9.3 Example for a closed dual path together with W shaded in grey. The implied A- or C-vacancies are in blue.

(south-east corner). For convenience call all other directed edges in Q_i^* *closed*. We call a dual path in Q_i^* connecting $D^{(2,*)}$ to $D^{(4,*)}$ *closed* iff all its edges are closed.

For $\omega \in \Omega$ consider the set W of vertices in $Q_i^{(v)} \setminus (D^{(2)} \cup D^{(4)})$ that are reachable by a B-traversable B-path (recall: up-right path) starting at $D^{(1)} \setminus (D^{(2)} \cup D^{(4)})$ and let the *contour* be the set of faces x^* that have a vertex inside and a vertex outside of $\{W \cup D^{(1)} \setminus (D^{(2)} \cup D^{(4)})\}$ incident to them. Not finding a B-traversable hard interior B-crossing on $Q_i^{(v)}$ then, by construction, implies that the contour is a closed dual path in Q_i^* connecting $D^{(2,*)}$ to $D^{(4,*)}$ (see Figure 9.3).

For a fixed $\omega \in \Omega$ let $\Gamma = \Gamma(\omega)$ be a closed non-backtracking dual path connecting the left to the right boundary and n_n, n_e, n_s, n_w be the amount of north, east, south and west steps in it respectively. Γ being closed then implies the existence of at least $(n_e + n_s)/2$ A- or C-vacancies, only half since if an east step follows a south step they have the same associated vertex. Further note that by construction of $Q_i^{(v)}$ every eighth step north an additional step east or south has to be made to reach the right boundary while any step west immediately implies another step east. So, Γ being closed implies the existence $\Theta(|\Gamma|)$ A- or C-vacancies[2]. Let Π_{x^*} be the set of dual paths starting at $x^* \in D^{(2,*)}$ and ending at $D^{(4,*)}$. We then have for some constants κ, C,

$$\begin{aligned}
\mu(\mathcal{A}^c) &\leq \sum_{x^* \in D^{(2,*)}} \sum_{\Gamma \in \Pi_{x^*}} \mu(\Gamma \text{ is closed}) \\
&\leq \sum_{x^* \in D^{(2,*)}} \sum_{\Gamma \in \Pi_{x^*}} (q_A + q_C)^{\Theta(|\Gamma|)} \\
&\leq \sum_{x^* \in D^{(2,*)}} \sum_{k=\kappa\ell}^{\infty} 3^k (q_A + q_C)^{\Theta(k)} \\
&\leq CN\ell 2^{-\ell}
\end{aligned}$$

[2]Note, we are not saying that there are only $\Theta(|\Gamma|)$ vacancies, but that the directly implied amount is of this order

where we chose q_B small enough and use that $q_A + q_C \to 0$ as $q_B \to 0$. The proof for horizontal strips is analogous and the claim follows. $\square$

With this we can calculate the failing probability of finding a B-traversable grid.

Corollary 9.13. *Let $\mathcal{E}^{(B,1)}$ be the event that*

- *there is a B-traversable B-grid,*
- *there is an intersection point $x_{i,j}$ in the above grid with $i, j > N/2$ such that there exists $\mathbf{e} \in \mathcal{B} = \{(0,1),(1,0)\}$ with $\omega_{x_{(i,j)+\mathbf{e}}} = B$,*

Then, for parameter sets such that $(q_A + q_C) \to 0$ as $q_B \to 0$ we have

$$\lim_{q_B \to 0} |\mathcal{Q}^{(B)}| \mu(1 - \mathbb{1}_{\mathcal{E}^{(B,1)}}) = 0.$$

Proof. Recall the event $\mathcal{A}$ from Lemma 9.12 that a strip is B-traversable. We can take a union bound to find a constant κ such that

$$\mu(\text{no } B\text{-traversable } B\text{-grid}) \leq 2(N+1)\mu(\mathcal{A}^c) \leq 2^{-\kappa\theta_B^{3/2}}$$

for q_B small enough. For a B-grid $\mathcal{C}$ we then have

$$\begin{aligned} &\mu(\text{no } i, j > N/2 \text{ such that } \exists \mathbf{e} \in \mathcal{B} : \omega_{x_{(i,j)+\mathbf{e}}} = B \,|\, \mathcal{C}\ B\text{-traversable}) \\ &\leq (1 - q_B)^{\Theta(N^2)} \leq e^{-\kappa\theta_B^2}. \end{aligned}$$

Using that $|\mathcal{Q}^{(B)}| = O((N\ell)^2)$ the claim follows. $\square$

This gives us a B vacancy on an intersection point and the necessary B-traversable paths to bring it into $Q_{0,0}$. The intersection point $x_{0,0}$ is still random though so we require another set of B-traversable paths to bring the B-vacancy to a deterministic point.

Lemma 9.14. *Let $\mathcal{E}^{(B,2)}$ be the event that the boundary $D_{0,0}^{(1)}$ is B-traversable. Then we have for parameter sets such that $\ell^2(q_A + q_C) \to 0$ as $q_B \to 0$ that*

$$\lim_{q_B \to 0} \mathrm{Supp}(\mathcal{E}^{(B,2)}) \mu(1 - \mathbb{1}_{\mathcal{E}^{(B,2)}}) = 0.$$

Proof. Follows immediately since

$$\mu(1 - \mathbb{1}_{\mathcal{E}^{(B,2)}}) = 1 - (1 - (q_A + q_C))^\ell \leq (q_A + q_C)\ell$$

and $\mathrm{Supp}(\mathcal{E}^{(B,2)}) = |D_{0,0}^{(1)}| = \ell$. $\square$

To show that $\mathcal{E}^{(B)} = \mathcal{E}^{(B,1)} \cap \mathcal{E}^{(B,2)}$ allows us to find an inequality like Equation (9.1) we need to introduce another tool.

Lemma 9.15 (Extending the variance)**.** *Let $\mathcal{A} = \mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{A}_3$ be an event on Ω, let $V_i := \mathrm{Supp}(\mathcal{A}_i)$ for $i \in [3]$ and assume that $V_i \cap V_j = \emptyset$ for any pair $i \neq j$. Then, for any $f \in L^2(\mu)$ and for the conditional variance $\mathrm{Var}_x(f \,|\, \mathcal{A}) = \mu_x(f^2 \,|\, \mathcal{A}) - (\mu_x(f \,|\, \mathcal{A}))^2$ we find*

$$\mu(\mathbb{1}_\mathcal{A} \mathrm{Var}_{V_1}(f \,|\, \mathcal{A})) \leq \mu(\mathbb{1}_\mathcal{A} \mathrm{Var}_V(f \,|\, \mathcal{A}))$$

for $V = V_1 \cup V_2$.

Remark 9.16. The usual use case is that we have an event $\mathcal{A}$ with a large support that we split into two smaller events $\mathcal{A}_1$, $\mathcal{A}_2$ and the 'rest' $\mathcal{A}_3$ which is why V only contains V_1 and V_2.

Proof. Write $\mathcal{A}' = \mathcal{A}_1 \cap \mathcal{A}_3$ and calculate directly

$$\begin{aligned}\mu(\mathbb{1}_{\mathcal{A}}\mathrm{Var}_{V_1}(f \mid \mathcal{A})) &= \mu_{V_2}(\mathcal{A}_2)\mu_{V_2^c}[\mathbb{1}_{\mathcal{A}'}\mu_{V_2}(\mathrm{Var}_{V_1}(f \mid \mathcal{A}) \mid \mathcal{A}_2)] \\ &= \mu_{V_2}(\mathcal{A}_2)\mu_{V_2^c}\Big[\mathbb{1}_{\mathcal{A}'}\mu_{V_2}(\mu_{V_1}(f^2 \mid \mathcal{A}) - (\mu_{V_1}(f \mid \mathcal{A}))^2 \mid \mathcal{A}_2)\Big] \\ &\le \mu_{V_2}(\mathcal{A}_2)\mu_{V_2^c}\Big[\mathbb{1}_{\mathcal{A}'}\Big(\mu_V(f^2 \mid \mathcal{A}) - (\mu_V(f \mid \mathcal{A}))^2\Big)\Big] \\ &= \mu[\mathbb{1}_{\mathcal{A}}\mathrm{Var}_V(f \mid \mathcal{A})],\end{aligned}$$

where in the first inequality we used Jensen's inequality and in the last equality we used that $\mathrm{Var}_V(f \mid \mathcal{A})$ does not depend on spins in V_2 anymore. □

Any configuration in $\mathcal{E}^{(B)}$ potentially contains many conforming B-grids so let us introduce a partial order on them. Let $\Gamma = (x^{(1)}, \dots, x^{(n)})$ and $\Gamma' = (y^{(1)}, \dots, y^{(m)})$ be two hard interior crossings of the same strip that cross in a single point $x^{(i)} = y^{(j)}$. If $x^{(i+1)} \prec y^{(j+1)}$ then we say that Γ is smaller than Γ'.

This generalises to a partial order on any family of hard interior crossings of the same strip with multiple crossing points if the above condition is fulfilled after *every* crossing point. Note that this is only a partial order but there is a unique smallest crossing. For $\omega \in \mathcal{E}^{(B)}$ we write $\mathcal{G}(\omega)$ for the B-grid with the smallest crossings in each strip conforming to $\mathcal{E}^{(B)}$.

A final remark about notation: We will write $\mu^{(h)}(\cdot) := \mu(\cdot \mid \{\star, h\})$ and $\mathrm{Var}^{(h)}(\cdot) := \mathrm{Var}(\cdot \mid \{\star, h\})$ for the measure resp. variance conditioned to be in the state space $\{\star, h\}$. Recall further that $\mathcal{Q}^{(B)}$ is the grid of base cells with parameters N, ℓ of which the smallest vertex in the $\prec$-partial order (i.e. the closest vertex to the origin) is $z_B := \mathbf{e}_1 + 3\mathbf{e}_2$.

Lemma 9.17. *Let $f \in L^2(\mu)$. For any $\varepsilon > 0$ we find a $q(\varepsilon) > 0$ such that*

$$\mu_{\mathcal{Q}^{(B)}}(\mathbb{1}_{\mathcal{E}^{(B)}}\mathrm{Var}_{z_B}^{(B)}(f)) \le 2^{\theta_B^2(1+\varepsilon)/4} \sum_{y \in \mathcal{Q}^{(B)}} \mu_{\mathcal{Q}^{(B)}}\Big[c_y^B q_B p(\nabla_y^{(B)} f)^2\Big],$$

for $q_B < q(\varepsilon)$.

Proof. For simplicity we write $\mu_{\mathcal{Q}^{(B)}} = \mu$ in this proof. There might be many intersection points $x_{i,j} \in X(\mathcal{G}(\omega))$ such that $\omega_{x_{(i,j)+\mathbf{e}}} = B$ for some $e \in \mathcal{B}$. Introduce the constraint $c_{x_{i,j}}^{(\mathcal{G})}$ that there is an $\mathbf{e} \in \mathcal{B}$ such that $\omega_{x_{(i,j)+\mathbf{e}}} = B$ and denote by $\xi(\omega)$ the vertex in $X(\mathcal{G}(\omega))$ with the highest coordinate in the lexicographic order such that $c_{\xi(\omega)}^{(\mathcal{G})} = 1$. Since this uniquely identifies a grid and an intersection point we have

$$\mu\Big[\mathbb{1}_{\mathcal{E}^{(B)}}\mathrm{Var}_{z_B}^{(B)}(f)\Big] = \sum_{\mathcal{C}\text{ grid}} \sum_{x \in X(\mathcal{C})} \mu\Big[\mathbb{1}_{\mathcal{G}=\mathcal{C}, \xi=x, \mathcal{E}^{(B,2)}}\mathrm{Var}_{z_B}^{(B)}(f)\Big].$$

Let us upper bound a generic summand $\mu\Big[\mathbb{1}_{\mathcal{G}=\mathcal{C}, \xi=x, \mathcal{E}^{(B,2)}}\mathrm{Var}_{z_B}^{(B)}(f)\Big]$ and assume without loss of generality that $x = x_{N,N-1}$. Let $V \subset X(\mathcal{C})$ be a subset with $x_{0,0}, x_{N,N-1} \in V$. The event $\mathcal{G} = \mathcal{C}$ on $V \cap z_B$ reduces to requiring B-traversability so that we can extend the variance (Lemma 9.15)

$$\mu\Big[\mathbb{1}_{\mathcal{G}=\mathcal{C}, \xi=x, \mathcal{E}^{(B,2)}}\mathrm{Var}_{z_B}^{(B)}(f)\Big] = \mu\Big[\mathbb{1}_{\mathcal{G}=\mathcal{C}, \xi=x, \mathcal{E}^{(B,2)}}\mathrm{Var}_{V \cup z_B}^{(B)}(f)\Big].$$

Using block relaxation (Lemma 6.17) we have

$$\mu\Big[\mathbb{1}_{\mathcal{G}=\mathcal{C}, \xi=x, \mathcal{E}^{(B,2)}}\mathrm{Var}_{V \cup z_B}^{(B)}(f)\Big] \le \frac{2}{q_B}\mu\Big[\mathbb{1}_{\mathcal{G}=\mathcal{C}, \xi=x, \mathcal{E}^{(B,2)}}\Big(\mathbb{1}_{\omega_{x_{0,0}}=B}\mathrm{Var}_{z_B}^{(B)}(f) + \mathrm{Var}_V^{(B)}(f)\Big)\Big]. \quad (9.2)$$

Let us deal with both these terms separately and start with the first summand. For any $\omega \in \mathcal{E}^{(B,2)} \cap \{\mathcal{G} = \mathcal{C}\}$ there is a unique shortest B-traversable B-path $\Gamma(\omega) \subset D_{0,0}^{(1)} \cup \mathcal{C}$ from z_B to $x_{0,0} - \mathbf{e}$ for some $\mathbf{e} \in \mathcal{B}$. As before, the event $\mathcal{E}^{(B,2)} \cap \{\mathcal{G} = \mathcal{C}\}$ on Γ simplifies to Γ being B-traversable. Thus, we can extend the variance again to get

$$\mu\Big[\mathbb{1}_{\mathcal{G}=\mathcal{C},\xi=x,\mathcal{E}^{(B,2)},\omega_{x_{0,0}}=B}\operatorname{Var}_{z_B}^{(B)}(f)\Big] \le \mu\Big[\mathbb{1}_{\mathcal{G}=\mathcal{C},\xi=x,\mathcal{E}^{(B,2)},\omega_{x_{0,0}}=B}\operatorname{Var}_{\Gamma}^{(B)}(f)\Big]. \tag{9.3}$$

Consider the auxiliary model on Γ with equilibrium measure $\mu_\Gamma^{(B)}$ that is given by the one-dimensional East model where B-vacancies are the vacancy state and any other state is the particle state and note that with $\omega_{x_{0,0}} = B$ this has ergodic boundary conditions. Since $|\Gamma| = O(\ell)$ we find a constant $\kappa > 0$ with Theorem 2.3 such that

$$\mathbb{1}_{\omega_{x_{0,0}}=B}\operatorname{Var}_{\Gamma}^{(B)}(f) \le 2^{\kappa\theta_B \log_2(\theta_B)} \sum_{y\in\Gamma} \mu_\Gamma^{(B)}\Big(c_x^B(1-q_B)q_B\big(\nabla_y^{(B)} f\big)^2\Big),$$

for q_B small enough, where we used that the one-dimensional constraints on Γ lower bound the two-dimensional constraints c_x^B for $x \in \Gamma$ and that $p > \Delta$ to bound the $1/(q_B + p)$ term coming from the conditional density of vacancies and particles in the East model. Inserting back into Equation (9.3) gives terms like

$$\sum_{y\in\Gamma} \mu\Big[\mathbb{1}_{\mathcal{G}=\mathcal{C},\xi=x,\mathcal{E}^{(B,2)}} c_y^B p q_B\big(\nabla_y^{(B)} f\big)^2\Big] \le \sum_{y\in Q_{0,0}} \mu\Big[\mathbb{1}_{\mathcal{G}=\mathcal{C},\xi=x,\mathcal{E}^{(B,2)}} c_y^B p q_B\big(\nabla_y^{(B)} f\big)^2\Big].$$

Contrary to Γ, $Q_{0,0}$ is not dependent on the specific $\mathcal{C}$ and ξ anymore so that we can resolve the sum over them to get that the first summand Equation (9.2) gives a contribution of

$$2^{\kappa\theta_B \log_2(\theta_B)} \sum_{y\in Q_{0,0}} \mu\Big[c_x^B q_B p\big(\nabla_y^{(B)} f\big)^2\Big].$$

For the second summand in Equation (9.2) note that we have not yet specified the subset $V \subset X(\mathcal{C})$. $X(\mathcal{C})$ is isomorphic to a $(0,1)$-squeezed box in $\mathbb{Z}^2$ and the dynamics with the constraints $c_y^{(\mathcal{G})}$ are equivalent to a two-dimensional East process on that box. Thus by Proposition 6.6(i) we find a subset $\{x_{0,0}, x_{N,N-1}\} \subset V \subset X(\mathcal{C})$ such that

$$\operatorname{Var}_V^{(B)}(f) \le 2^{\theta_B^2(1+\varepsilon/2)/4} \sum_{y\in V} \mu_V^{(B)}\Big[c_y^{(\mathcal{G})} q_B p\big(\nabla_y^{(B)} f\big)^2\Big]$$

for q_B small enough. Given the events $\{\mathcal{G} = \mathcal{C}\} \cap \{\xi = x\}$ we can again extend to B-traversable paths this time between points on $X(\mathcal{C})$ and using completely analogous calculations to the first summand we get.

$$\begin{aligned}\mu\Big[\mathbb{1}_{\mathcal{G}=\mathcal{C},\xi=x,\mathcal{E}^{(B,2)}}\operatorname{Var}_V^{(B)}(f)\Big] &\le 2^{\kappa\theta_B \log_2(\theta_B)} \sum_{y\in\mathcal{C}} \mu\Big[\mathbb{1}_{\mathcal{G}=\mathcal{C},\xi=x,\mathcal{E}^{(B,2)}} c_y^B q_B p\big(\nabla_y^{(B)} f\big)^2\Big] \\ &\le 2^{\kappa\theta_B \log_2(\theta_B)} \sum_{y\in\mathcal{Q}^{(B)}} \mu\Big[\mathbb{1}_{\mathcal{G}=\mathcal{C},\xi=x,\mathcal{E}^{(B,2)}} c_y^B q_B p\big(\nabla_y^{(B)} f\big)^2\Big].\end{aligned}$$

Resolve the sum over $\mathcal{C}$ and ξ again and note that $Q_{0,0}$ is counted twice leading to an additional term of the order $O(\ell^2)$ that we absorb into κ to get the claim. □

Remark 9.18. For simplicity we limited the discussion in this section to B-grids. The results generalise to h-grids with q_h going to 0 and the conditions for $q_A + q_C$ are substituted with conditions for $1 - q_h - p$.

Notice that there are multiple ways to choose ℓ and still have this proof work. In fact, any $\ell = 2^\alpha$ with for example $\alpha \in (\theta_B + 10\log_2(\theta_B), \theta_B^{2-\varepsilon}/\log_2(\theta_B))$ gives analogous results for correspondingly adapted conditions on q_A and q_C. As that would have complicated the exposition for little gain in generality of the result, in the sense that Theorem 6 would still not hold for any parameter set $\mathbf{q}$, we chose to limit the exposition in this thesis to $\ell = \lfloor \theta_{q_{\min}}^{3/2} \rfloor$.

9.2 Low vacancy density: Proof of Theorem 6(3.i)

Fix an $\varepsilon > 0$, let $\mathbf{q}$ be a parameter set such that $\min_{h\in G} q_h = q_B$ and $(q_A + q_C)\theta_B^3 \to 0$. By Lemma 8.4 we have $\gamma(G, \mathbf{q}) \le \gamma_2(q_B)$, so using Theorem 2.3 we need to show that there is a $\delta > 0$ so that for $q_B < \delta$ we have

$$\gamma(G, \mathbf{q}) \ge 2^{-\theta_B^2(1+\varepsilon)/4}.$$

For $h \in G$ let $\mathcal{E}^{(h)} = \mathcal{E}^{(h,1)} \cap \mathcal{E}^{(h,2)}$ be the events from Section 9.1, let $\mathcal{E}_x^{(h)}$ be the correspondingly translated event and let z_h be the analogous vertices to z_B. Using the results from Section 9.1 we can get h-vacancies to z_h. We thus need an event that allows us to bring the vacancies back to the origin.

Let $\mathcal{E}_x^{(0)}$ be the event that there is no vacancy on $\{x + i\mathbf{e}_1 : i \in [3]\} \cup \{x - i\mathbf{e}_2 : i \in [3]\}$ and $\mathcal{E}_x := \mathcal{E}_x^{(0)} \cap \bigcap_{h\in G} \mathcal{E}_x^{(h)}$. By construction the family $\{\mathcal{E}_x\}_{x\in\mathbb{Z}^2}$ satisfies the exterior condition with respect to the exhausting and increasing family of sets $\{V_n\}_{n\in\mathbb{Z}}$ given in Lemma 9.11. By assumption on the parameter set and Corollary 9.13 and Lemma 9.14, Equation (8.1) holds for the family $\{\mathcal{E}_x\}_{x\in\mathbb{Z}^2}$ for q_B small enough. Thus, we can apply the exterior condition theorem, Theorem 8.2, and Lemma 8.5 to get

$$\operatorname{Var}(f) \le 4 \sum_{x\in\mathbb{Z}^2} \mu(\mathbb{1}_{\mathcal{E}_x} \operatorname{Var}_x(f)) \le C \sum_{x\in\mathbb{Z}^2} \sum_{h\in G} \mu(\mathbb{1}_{\mathcal{E}_x^{(0)} \cap \mathcal{E}_x^{(h)}} \operatorname{Var}_x^{(h)}(f)). \tag{9.4}$$

Let us consider w.l.o.g. only the term for $h = B$ and $x = 0$ and leave away the subscript x. Recall that we write $z_B = \mathbf{e}_1 + 3\mathbf{e}_2$ which by $\mathcal{E}^{(B,2)}$ is B-traversable so that we can extend the variance, Lemma 9.15, and apply the block relaxation Lemma, Lemma 6.17:

$$\begin{aligned}
\mu(\mathbb{1}_{\mathcal{E}^{(0)}\cap\mathcal{E}^{(B)}} \operatorname{Var}_0^{(B)}(f)) &\le \mu(\mathbb{1}_{\mathcal{E}^{(0)}\cap\mathcal{E}^{(B)}} \operatorname{Var}_{\{0,z_B\}}^{(B)}(f)) \\
&\le \frac{C}{q_B} \mu\Big[\mathbb{1}_{\mathcal{E}^{(0)}\cap\mathcal{E}^{(B)}} \mu_0^{(B)}\Big(\mathbb{1}_{\omega_{z_B}=B} \operatorname{Var}_0^{(B)}(f) + \operatorname{Var}_{z_B}^{(B)}(f)\Big)\Big] \\
&= \frac{C}{q_B} \mu\Big[\mathbb{1}_{\mathcal{E}^{(0)}\cap\mathcal{E}^{(B)}} \Big(\mathbb{1}_{\omega_{z_B}=B} \operatorname{Var}_0^{(B)}(f) + \operatorname{Var}_{z_B}^{(B)}(f)\Big)\Big],
\end{aligned}$$

where in the last equality we used that $\operatorname{Supp}(\mathcal{E}) \cap \{0\} = \emptyset$ and the tower property. By Lemma 9.17 we can upper bound the second summand by

$$\mu\Big(\mathbb{1}_{\mathcal{E}^{(0)}\cap\mathcal{E}^{(B)}} \operatorname{Var}_{z_B}^{(B)}(f)\Big) \le 2^{\theta_B^2(1+\varepsilon)/4} \mu\Big(\mathcal{D}_{\mathcal{Q}^{(B)}}(f)\Big),$$

where we added the missing A- and C-transition terms to get a contribution to the Dirichlet form. For the first summand let $\Gamma = \{0, \mathbf{e}_2, \dots, 3\mathbf{e}_2\}$. Consider the auxiliary model on Γ with B-vacancies as the good state and non-B-vacancy states as bad states with one-dimensional East model constraints. This model has good boundary conditions if $\omega_{z_B} = B$. Use that $\mathcal{E}^{(0)}$ on Γ reduces to requiring B-traversability and use Theorem 2.3 to get a constant $\kappa > 0$ such that

$$\mu(\mathbb{1}_{\mathcal{E}^{(0)}, \mathcal{E}^{(B)}, \{\omega_{z_B}=B\}} \operatorname{Var}_0^{(B)}(f)) \le \mu(\mathbb{1}_{\mathcal{E}^{(0)}, \mathcal{E}^{(B)}, \{\omega_{z_B}=B\}} \operatorname{Var}_\Gamma^{(B)}(f))$$

$$\leq 2^{\kappa\theta_B \log_2(\theta_B)} \sum_{y\in\Gamma} \mu\Big(c_y^B p q_B (\nabla_y^{(B)} f)^2\Big)$$
$$\leq 2^{\kappa\theta_B \log_2(\theta_B)} \mu(\mathcal{D}_\Gamma(f))$$

for q_B small enough. By translation invariance we get analogous terms for any $x \in \mathbb{Z}^2$. When taking the sum over x we need to account for the overcounting, which we recall is how many times a single vertex $y \in \mathbb{Z}^2$ appears in the various Dirichlet forms that we get in the above way for the different $x \in \mathbb{Z}^2$. In this case, for any $y \in \mathbb{Z}^2$ there are $O(|\mathcal{Q}_B|)$ different $x \in \mathbb{Z}^2$ such that $y \in \mathcal{Q}_x^{(B)}$ we can absorb[3] the overcounting into ε for q_B small enough. Thus for $h = B$ the r.h.s. in Equation (9.4) is upper bounded by

$$\sum_{x\in\mathbb{Z}^2} \mu(\mathbb{1}_{\mathcal{E}^{(0)}\cap\mathcal{E}^{(B)}} \mathrm{Var}_0^{(B)}(f)) \leq 2^{\theta_B^2(1+\varepsilon)/4} \mathcal{D}(f),$$

for q_B small enough. The calculation works analogously for each $h \in \mathcal{G}$ and we get the claim for the chosen $\mathbf{q}$ by arbitrariness of ε. Further, the proof also works analogously for any $\mathbf{q}$ such that $q_{\max}\theta_{q_{\min}}^3 \to 0$ as $q_{\min} \to 0$ so that we have part (3.i) of Theorem 6. □

9.3 Single frequent vacancy type: Proof of Theorem 6(3.ii)

Throughout this section assume that $\mathbf{q}$ is a parameter set such that $q_{\min} = q_B$, $q_{\max}\theta_{q_{\text{med}}}^3 / \log_2(\theta_B) \to \infty$ and $q_{\text{med}}\theta_B^6 \to 0$ as $q_B \to 0$ where we recall that q_{med} is the remaining element of $\mathbf{q} \setminus \{q_{\max}, q_{\min}\}$.

In this case the above results do not apply anymore. We resolve this problem by working on boxes and defining traversable configurations on them that do not exclude the frequent vacancy type. We then show that on this coarse grained lattice we can apply the results from Section 9.1 again and conclude the proof by using auxiliary models and the path method to go from the coarse grained lattice back to $\mathbb{Z}^2$.

We start with the proofs for the case where $q_{\min} = q_B$. We will see later that this is sufficient as the proofs for $q_{\min} \in \{q_A, q_C\}$ are analogous.

9.3.1 The case $q_{\max} = q_A$

Assume for this subsection that $q_{\max} = q_A$ and $q_{\text{med}} = q_C$. We start by defining the coarse graining and the states on the coarse-grained lattice.

Definition 9.19. For $\mathbf{j} \in \mathbb{Z}^2$ and $L = \lfloor \theta_B^3 \rfloor$ let $\Lambda_{\mathbf{j}}$ be an equilateral box of side length $L-1$ and origin $(L+1)\mathbf{j}$ and let $W_{\mathbf{j}}$ be the outline of it, i.e. the shortest cycle containing $(L+1)\mathbf{j} + \{0, (L-1)\mathbf{e}_1, (L-1)\mathbf{e}_2, (L-1)(\mathbf{e}_1 + \mathbf{e}_2)\}$. Let the *enlargement* $EW_{\mathbf{j}}$ *of* $W_{\mathbf{j}}$ be the union of $W_{\mathbf{j}}$ with the set $\tilde{\Lambda} \setminus \Lambda_{\mathbf{j}}$ where $\tilde{\Lambda}$ is an equilateral box of side length L, origin $(L+1)\mathbf{j}$ and denote the top right corner of $EW_{\mathbf{j}}$ by $x_{\mathbf{j}} = (L+1)\mathbf{j} + L(\mathbf{e}_1 + \mathbf{e}_2)$. For a $\omega \in \Omega$ we call $EW_{\mathbf{j}}$

- *B-traversable* (see Figure 9.4) if
  - $\omega_x \in \{\star, A\}$ for any $x \in W_{\mathbf{j}}$,
  - $\omega_x \in \{\star, A, B\}$ for any $x \in EW_{\mathbf{j}} \setminus W_{\mathbf{j}}$ and
  - for any $i \in [2]$ there is at least one $x \in (L+1)\mathbf{j} + \{\mathbf{e}_i, 2\mathbf{e}_i, \ldots, (L-1)\mathbf{e}_i\}$ (i.e. parts of the left respectively bottom boundary of $W_{\mathbf{j}}$) such that $\omega_x = A$.

[3] We often use the term absorb in this context, where we either mean make the constant larger/smaller or here specifically, where ε is fixed, do the whole proof for $\varepsilon/2$ and only in the final step write ε upper bounding any lower order term by $2^{\theta_B^2\varepsilon/4}$.

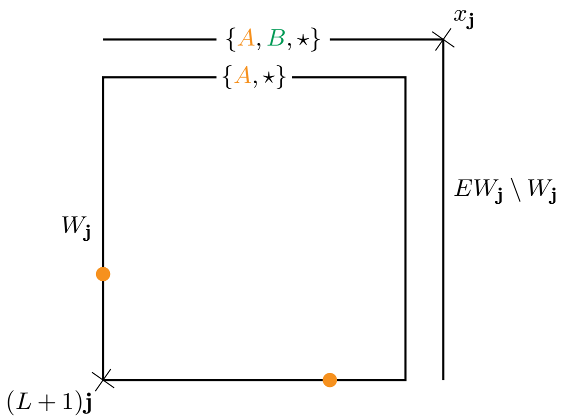

FIGURE 9.4 Illustration of a B-traversable $EW_{\mathbf{j}}$. Note that the A-vacancies on the bottom and left boundary can be anywhere on that boundary.

Let $q_{BT} := \mu_{EW_{\mathbf{j}}}(B\text{-traversable})$.

- B-*super* if $EW_{\mathbf{j}}$ is B-traversable and $\omega_{x_{\mathbf{j}}} = B$. Let $q_{BS} := \mu_{EW_{\mathbf{j}}}(B\text{-super})$.
- B-*evil*[4] if it is not B-traversable. Let $q_{BE} := \mu_{EW_{\mathbf{j}}}(B\text{-evil})$.

Attention: Previously, if we said that $EW_{\mathbf{j}}$ was B-traversable, we meant that $\omega_x \in \{\star, B\}$ for any $x \in EW_{\mathbf{j}}$ instead of the above definition. In the context in which $q_A \gg q_B$ this notion of B-traversability has a very small equilibrium probability so it is not useful for the proof of part (3.ii). We justify the recycling of the name since the two notions of traversability play analogous roles. In Section 9.1 we looked for grids of paths with vertices only in $\{\star, B\}$. In this section we look for grids where each vertex is a $EW_{\mathbf{j}}$ that is B-traversable in the above sense.

To be able to use the results from Section 9.1 on the coarse-grained lattice we need to show that B-super boxes play the role of B-vacancies and B-evil boxes the role of A and C vacancies.

Lemma 9.20. *For $EW_{\mathbf{j}}$ as in Definition 9.19 we have*

$$\frac{\theta_{q_{BS}}}{\theta_B} \to 1, \qquad \theta^3_{q_{BS}} q_{BE} \to 0,$$

as $q_B \to 0$.

Proof. It is immediate to see that the right limit implies the left one. Indeed, if $\theta^3_{q_{BS}} q_{BE} \to 0$ then in particular $q_{BE} \to 0$ so that $q_{BT} \to 1$ which implies

$$\frac{q_{BS}}{q_B} \geq \frac{q_{BT}}{(1-q_C)^2} \to 1.$$

Let us come to the right limit. A union bound gives for q_B small enough

$$\begin{aligned} q_{BE} &\leq 1 - (p+q_A)^{4(L-1)} + 1 - (1-q_C)^{2L-1} + 1 - \left(1 - (p/(p+q_A))^{L-1}\right)^2 \\ &\leq O(L(q_B+q_C)) + 2(p/(p+q_A))^{L-1} \\ &\leq O(L(q_B+q_C)) + 2/(1+5\log_2(\theta_B)/\theta_B^3)^{L-1}, \end{aligned}$$

so that $q_{BE}\theta^3_{q_{BS}} \leq q_{BE}\theta^3_B \to 0$ for $q_B \to 0$. □

[4]We use super and evil instead of the more common good and bad to avoid confusion in the notation with $G \subset H_d$ and B-vacancies.

Remark 9.21. While $q_A\theta_B^3 \to \infty$ as $q_B \to 0$ we thus find a renormalisation such that we again have the equivalent of $(q_A+q_C)\theta_B^3 \to 0$ from Section 9.1 on the renormalised lattice.

For $\mathbf{j} \in \mathbb{Z}^2$ let $\Omega_{\mathbf{j}}^* = S(G)^{EW_{\mathbf{j}}}$, $\mu_{\mathbf{j}}^*(\cdot) = \mu_{EW_{\mathbf{j}}}(\cdot \mid B\text{-traversable})$ and let $\mathrm{Var}_{\mathbf{j}}^*(f)$ be the associated variance. We will only use the letter $\mathbf{j}$ in bold font to refer to indices of $EW_{\mathbf{j}}$ and thus say interchangeably that $\mathbf{j}$ or $EW_{\mathbf{j}}$ is B-traversable, B-super or B-evil. Let us come to the analogue statement of Lemma 9.17, for which we need to define the analogue of the events $\mathcal{E}^{(B,i)}$ for the lattice of boxes. We define $\mathcal{Q}^{(B,*)} = \{EW_{\mathbf{j}} \colon \mathbf{j} \in \mathcal{Q}^{(B)}\}$ for $\mathcal{Q}^{(B)}$ with side length $\ell = \lceil \theta_{q_{BS}}^{3/2} \rceil$ and square side length $N = 2^{\lceil \theta_{q_{BS}}/2 + \log_2(\theta_{q_{BS}}) \rceil}$. The vector $z_B = \mathbf{e}_1 + 3\mathbf{e}_2$ we now write as $\mathbf{j}_B$.

Let $\mathcal{E}^{(B,1,*)}$ be the event that we find a B-grid $\mathcal{C}$ in $\mathcal{Q}^{(B)}$ such that $EW_{\mathbf{j}}$ is B-traversable for any $\mathbf{j} \in \mathcal{C}$ and such that there is an intersection point $\mathbf{j}_{i,j} \in X(\mathcal{C})$ with $i, j > N/2$ and $\mathbf{e} \in \mathcal{B}$ such that $EW_{\mathbf{j}_{(i,j)+\mathbf{e}}}$ is B-super.

Let $\mathcal{E}^{(B,2,*)}$ be event that for each $\mathbf{j}$ on the boundary $D_{0,0}^{(1)}$, $EW_{\mathbf{j}}$ is B-traversable. We write $\mathcal{E}^{(B,*)} = \mathcal{E}^{(B,1,*)} \cap \mathcal{E}^{(B,2,*)}$. The support is included in $\mathcal{Q}^{(B,*)}$, i.e. $\mathrm{Supp}(\mathcal{E}^{(B,*)}) \subset \mathcal{Q}^{(B,*)}$ and $\mathcal{E}^{(B,*)}$ satisfies the exterior condition with respect to the same $\{V_n\}_{n\in\mathbb{Z}}$ as in Lemma 9.11.

The auxiliary model for which we state the analogue of Lemma 9.17 is given by the constraints $c_{\mathbf{j}}^{*,B}$ defined as the indicator over the event that there exists an $\mathbf{e} \in \mathcal{B}$ such that $\mathbf{j}+\mathbf{e}$ is B-super. Analogous to the $\mu^{(h)}$ notation we write $\mu^{(AB)}(\cdot) = \mu(\cdot \mid \{\star, A, B\})$ and $\mathrm{Var}^{(AB)}(f) := \mathrm{Var}(f \mid \{\star, A, B\})$.

Corollary 9.22. *For any $\varepsilon > 0$ we find a $\delta > 0$ such that*

$$\mu(\mathbb{1}_{\mathcal{E}^{(B,*)}} \mathrm{Var}_{x_{\mathbf{j}_B}}^{(AB)}(f)) \le 2^{\theta_B^2(1+\varepsilon)/4} \sum_{\mathbf{j}\in\mathcal{Q}^{(B)}} \mu\Big[\mathbb{1}_{\mathbf{j}\ B\text{-traversable}} c_{\mathbf{j}}^{*,B} \mathrm{Var}_{x_{\mathbf{j}}}^{(AB)}(f)\Big]$$

for $q_B < \delta$.

Remark 9.23. Normally when defining the East model on block lattices (see for example [13]) instead of Corollary 9.22 we expect statements of the form

$$\mu_{\mathcal{Q}^{(B)}}^*(\mathbb{1}_{\mathcal{E}^{(B,*)}} \mathrm{Var}_{\mathbf{j}_B}^*(f)) \le 2^{\theta_B^2(1+\varepsilon)/4} \sum_{\mathbf{j}\in\mathcal{Q}^{(B)}} \mu_{\mathcal{Q}^{(B)}}^*\Big[c_{\mathbf{j}}^{*,B} \mathrm{Var}_{\mathbf{j}}^*(f)\Big].$$

To upper bound the r.h.s. with a Dirichlet form of the ABC-model we need to be able to relax an entire B-traversable $EW_{\mathbf{j}}$ given that a neighbour is B-super, but this is not possible since we do not have an A-vacancy that can reach all of $W_{\mathbf{j}}$. The set $EW_{\mathbf{j}} \setminus W_{\mathbf{j}}$ on the other hand can be fully relaxed which is why we limit ourselves to the top-right points $x_{\mathbf{j}} \in EW_{\mathbf{j}} \setminus W_{\mathbf{j}}$.

Proof. As in the proof of Lemma 9.17 let $\mathcal{G}$ be the smallest B-grid with B-traversable crossings and ξ the vertex with the highest coordinate in the $\prec$-partial order such that if $\mathbf{j}_{i,j} = \xi$ then there is an $\mathbf{e} \in \mathcal{B}$ with a B-vacancy on $x_{\mathbf{j}_{(i,j)+\mathbf{e}}}$. Then,

$$\mu(\mathbb{1}_{\mathcal{E}^{(B,*)}} \mathrm{Var}_{x_{\mathbf{j}_B}}^{(AB)}(f)) = \sum_{\mathcal{C}\ B\text{-grid}} \sum_{\mathbf{j}'\in X(\mathcal{C})} \mu\Big[\mathbb{1}_{\mathcal{G}=\mathcal{C}, \xi=\mathbf{j}', \mathcal{E}^{(B,2,*)}} \mathrm{Var}_{x_{\mathbf{j}_B}}^{(AB)}(f)\Big].$$

To save some space let us write $\tilde{\mathcal{E}} := \{\mathcal{G} = \mathcal{C}\} \cap \{\xi = \mathbf{j}'\} \cap \mathcal{E}^{(B,2,*)}$. We upper bound a generic summand so fix a $\mathcal{C}$ and a $\mathbf{j}'$. Consider the subset $\mathcal{C}^{(TR,*)} := \{x_{\mathbf{j}} \colon \mathbf{j} \in \mathcal{C}\}$ of top right corners of $EW_{\mathbf{j}}$. The event $\{\mathcal{G} = \mathcal{C}\}$ reduces to $\omega_{x_{\mathbf{j}}} \in \{A, B, \star\}$ on any $x_{\mathbf{j}} \in \mathcal{C}^{(TR,*)}$. In an analogous proof to Lemma 9.17 we find

$$\mu\Big[\mathbb{1}_{\tilde{\mathcal{E}}} \mathrm{Var}_{x_{\mathbf{j}_B}}^{(AB)}(f)\Big] \le 2^{\theta_B^2(1+\varepsilon)/4} \sum_{y\in\mathcal{C}^{(TR,*)}} \mu\Big[\mathbb{1}_{\tilde{\mathcal{E}}} c_y^{(\mathcal{G})} \mathrm{Var}_y^{(AB)}(f)\Big]$$

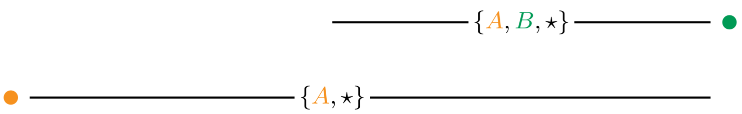

FIGURE 9.5 Starting situation from Lemma 9.24.

for q_B small enough where $c^{(\mathcal{G})}_{x_{\mathbf{j}}}$ is the constraint that there is an $\mathbf{e} \in \mathcal{B}$ such that $\mathbf{j} + \mathbf{e} \in \mathcal{G}$ and $x_{\mathbf{j}+\mathbf{e}}$ has a B-vacancy. Given $\tilde{\mathcal{E}}$ any $EW_{\mathbf{j}}$ for $\mathbf{j} \in \mathcal{G}$ is B-traversable so that $\mathbb{1}_{\tilde{\mathcal{E}}} c^{(\mathcal{G})}_y \leq \mathbb{1}_{\tilde{\mathcal{E}}} c^{*,B}_{\mathbf{j}}$. We can thus upper bound the sum in the r.h.s. by

$$\sum_{y \in \mathcal{C}^{(TR,*)}} \mu\Big[\mathbb{1}_{\tilde{\mathcal{E}}} c^{(\mathcal{G})}_y \mathrm{Var}^{(AB)}_y(f)\Big] \leq \sum_{\mathbf{j} \in \mathcal{Q}^{(B,*)}} \mu\Big[\mathbb{1}_{\tilde{\mathcal{E}},\mathbf{j}\ B\text{-traversable}} c^{*,B}_{\mathbf{j}} \mathrm{Var}^{(AB)}_{x_{\mathbf{j}}}(f)\Big].$$

We get the claim after resolving the sum over $\mathcal{G}$ and ξ and taking into account the overcounting which we can absorb into the ε. □

Given a B-traversable box with a neighbouring B-super box we want to recover from a generic term in the r.h.s. in Corollary 9.22 a Dirichlet form of the ABC-model. To that end, let us isolate two generic situations first. The first explains how to use the A-vacancies on $W_{\mathbf{j}}$ to relax $EW_{\mathbf{j}}$.

Lemma 9.24. *Let $C_2, C_1 = O(\theta_B^3)$ be two constants and consider two paths*

$$\Gamma_1 = \{0, \mathbf{e}_1, \dots, C_1\mathbf{e}_1\},$$
$$\Gamma_2 = \{-C_2\mathbf{e}_1 - \mathbf{e}_2, -(C_2 - 1)\mathbf{e}_1 - \mathbf{e}_2, \dots, C_1\mathbf{e}_1 - \mathbf{e}_2\}.$$

On these paths define the event $\mathcal{A}$ that on Γ_1 we find no C-vacancies, on Γ_2 no B- or C-vacancies, there is an A vacancy on $\Gamma_2 \setminus (\Gamma_1 - \mathbf{e}_2)$ and $\omega_{(C_1+1)\mathbf{e}_1} = B$ (see Figure 9.5). Then we find a constant $\kappa > 0$ such that for any $y \in \Gamma_1$

$$\mu(\mathbb{1}_{\mathcal{A}} \mathrm{Var}^{(AB)}_y(f)) \leq 2^{\kappa\theta_B \log_2(\theta_B)} \mu(\mathbb{1}_{\mathcal{A}} \mathcal{D}_{\Gamma_1 \cup \Gamma_2}(f)).$$

Proof. Consider the auxiliary model on Γ_1 with constraints c^B_x that samples from $\mu^{(AB)}$ at a legal ring. If the starting state is in $\mathcal{A}$ then any later state is as well and the spectral gap of the auxiliary process agrees with that of a one-dimensional East model with good boundary conditions. Thus, we can extend the variance (Lemma 9.15) and find a constant κ with Theorem 2.3 such that

$$\mu(\mathbb{1}_{\mathcal{A}} \mathrm{Var}^{(AB)}_y(f)) \leq \mu(\mathbb{1}_{\mathcal{A}} \mathrm{Var}^{(AB)}_{\Gamma_1}(f)) \leq 2^{\kappa\theta_B \log_2(\theta_B)} \sum_{x \in \Gamma_1} \mu(\mathbb{1}_{\mathcal{A}} c^B_x \mathrm{Var}^{(AB)}_x(f)),$$

for q_B small enough. For each x we can extend the variance and use block relaxation, Lemma 6.17, to get

$$\mu(\mathbb{1}_{\mathcal{A}} c^B_x \mathrm{Var}^{(AB)}_x(f)) \leq \frac{C}{q_A} \mu\Big[\mathbb{1}_{\mathcal{A}} c^B_x (\mathbb{1}_{\omega_{x-\mathbf{e}_2}=A} \mathrm{Var}^{(AB)}_x(f) + \mathrm{Var}^{(A)}_{x-\mathbf{e}_2}(f))\Big],$$

for q_B small enough. For the first summand we can write the variance as transition terms (Lemma 8.5) and use that $\mathbb{1}_{\omega_{x-\mathbf{e}_2}=A} \leq c^A_x$ to recover a term of the Dirichlet form. For the second summand we can use the enlargement trick (Lemma 6.10) so that

$$\mu(\mathbb{1}_{\mathcal{A}} \mathrm{Var}^{(AB)}_y(f)) \leq 2^{\kappa\theta_B \log_2(\theta_B)} \sum_{x \in \Gamma_1} \mu(\mathcal{D}_{\{x\} \cup \Gamma_2}).$$

The overcounting is of order $O(\theta_B^3)$ and can thus be absorbed into the κ and we recover the claim. □

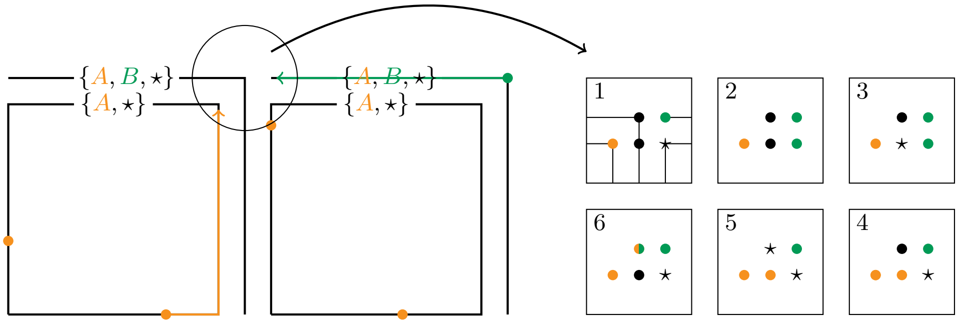

FIGURE 9.6 Left: Two enlarged boxes next to each other, the left is B-traversable, the right B-super. Right: The sequence of legal moves that brings B from a neighbouring B-super box to a B-traversable box, or alternatively removes it, indicated by the two-coloured node.

Being able to relax $EW_{\mathbf{j}}$ means that we can move the B-vacancy freely on it using the block relaxation Lemma. The second of our isolated results moves the B-vacancy from a neighbouring B-super box to a B-traversable box. This requires the path method since $x_{\mathbf{j}}$ does not neighbour a vertex in $W_{\mathbf{j}}$ and so we can not use Lemma 9.24 together with the block relaxation Lemma to move a B-vacancy here.

Lemma 9.25. *Consider the set $V = \{0, -\mathbf{e}_2, -\mathbf{e}_1 - \mathbf{e}_2\}$ and the event $\mathcal{A}$ given by the $\omega \in \Omega$ such that $\omega_{-\mathbf{e}_1-\mathbf{e}_2} = A$, $\omega_{\mathbf{e}_1-\mathbf{e}_2} = \star$ and $\omega_{\mathbf{e}_1} = B$. Further define the event $\mathcal{A}'$ given by the configurations ω such that $\omega_x \in \{\star, A, B\}$ for $x \in \{-\mathbf{e}_2, 0, \mathbf{e}_1\}$ and $\omega_x \in \{\star, A\}$ for $x \in \{-\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_1 - \mathbf{e}_2\}$. Then, for q_B small enough we find a constant κ such that*

$$\mu_V\left[\mathbb{1}_{\mathcal{A}}\mathrm{Var}_0^{(AB)}(f)\,|\,\mathcal{A}'\right] \le 2^{\kappa\theta_B}\sum_{x\in V}\mathcal{D}_V(f).$$

Proof. The proof is straightforward when looking at Figure 9.6. The vertex $\mathbf{e}_1 - \mathbf{e}_2$ starts in the neutral state and $\mathbf{e}_1$ has a B-vacancy. So we can put a B-vacancy on $\mathbf{e}_1 - \mathbf{e}_2$ (Figure 9.6(2)). This means that $-\mathbf{e}_2$ neighbours an A- and a B-vacancy so we can put $\star$ on it (Figure 9.6(3)), in particular we can put an A-vacancy (Figure 9.6(4)) so that the origin neighbours an A- or a B-vacancy and we can put it into any state. Notice that all these transitions are independent from the state of the origin so for any $\omega \in \mathcal{A}$ and any σ, that agrees with ω outside of the origin, we find a legal path of constant length connecting them and conclude with the path method. ☐

Armed with these results we can upper bound the right hand side in Corollary 9.22. For this we introduce the notation $EW(V) = \cup_{\mathbf{j}\in V} EW_{\mathbf{j}}$ for any subset $V \subset \mathbb{Z}^2$.

Lemma 9.26. *Let $\mathbf{j} \in \mathbb{Z}^2$ and $V = \{\mathbf{j}, \mathbf{j}+\mathbf{e}_1, \mathbf{j}+\mathbf{e}_2\}$. We find a constant $\kappa > 0$ such that*

$$\mu\left[\mathbb{1}_{\mathbf{j}\ B\text{-traversable}}c_{\mathbf{j}}^{*,B}\mathrm{Var}_{x_{\mathbf{j}}}^{(AB)}(f)\right] \le 2^{\kappa\theta_B\log_2(\theta_B)}\mathcal{D}_{EW(V)}(f)$$

for q_B small enough.

Proof. W.l.o.g. consider only the case $\mathbf{j} = \mathbf{0}$ and where the constraint on the l.h.s. of the claim is replaced by $\tilde{c} = \mathbb{1}_{EW_{\mathbf{e}_1}\text{ is }B\text{-super}}$. Let $U = x_{\mathbf{0}} + \{0, \mathbf{e}_1, -\mathbf{e}_2 - \mathbf{e}_1, -\mathbf{e}_2, -\mathbf{e}_2 + \mathbf{e}_1\}$ and let $\mathcal{A}$ be the event from Lemma 9.25 translated by $x_{\mathbf{0}}$, so that $\mathrm{Supp}(\mathcal{A}) \subset U$. Analogously define $\mathcal{A}'$ as the translated version of $\mathcal{A}'$ from Lemma 9.25. The event that $EW_{\mathbf{0}}$ and $EW_{\mathbf{e}_1}$ are B-traversable on U reduces to $\mathcal{A}'$. Thus, we can extend the variance, Lemma 9.15, and use the block relaxation Lemma 6.17,

$$\mu\left[\tilde{c}\mathbb{1}_{EW_{\mathbf{0}}\ B\text{-traversable}}\mathrm{Var}_{x_{\mathbf{0}}}^{(AB)}(f)\right] \le \mu[\tilde{c}\mathbb{1}_{EW_{\mathbf{0}}\ B\text{-traversable}}\mathrm{Var}_U(f\,|\,\mathcal{A}')]$$

$$\leq 2^{\kappa\theta_B}\mu\Big[\tilde{c}\mathbb{1}_{EW_{\mathbf{0}}\ B\text{-traversable}}\Big(\mathbb{1}_{\mathcal{A}}\mathrm{Var}^{(AB)}_{x_{\mathbf{0}}}(f)+\mathrm{Var}_{U\setminus\{x_{\mathbf{0}}\}}(f\,|\,\mathcal{A}')\Big)\Big]$$

for q_B small enough. The first summand can be upper bounded using Lemma 9.25. For the second summand write

$$\mu\Big[\tilde{c}\mathbb{1}_{EW_0\ B\text{-traversable}}\mathrm{Var}_{U\setminus\{x_{\mathbf{0}}\}}(f\,|\,\mathcal{A}')\Big]\leq\mu\left[\tilde{c}\mathbb{1}_{EW_0\ B\text{-traversable}}\sum_{y\in U\setminus\{x_{\mathbf{0}}\}}\mathrm{Var}_y(f\,|\,\mathcal{A}')\right].$$

For $y\in(EW_{\mathbf{0}}\setminus W_{\mathbf{0}})\cup(EW_{\mathbf{e}_1}\setminus W_{\mathbf{e}_1})$ we can use Lemma 9.24 and for the others we can use the enlargement trick (Lemma 6.10) to get the claim. □

Combining the previous results we thus have

$$\mu(\mathbb{1}_{\mathcal{E}^{(B,*)}}\mathrm{Var}^{(AB)}_{x_{\mathbf{j}_B}}(f))\leq 2^{\theta_B^2(1+\varepsilon)/4}\mu\Big[\mathcal{D}_{\mathcal{Q}^{(B,*)}}(f)\Big],\tag{9.5}$$

for q_B small enough. For C-vacancies we can use the same construction of $EW_{\mathbf{j}}$ and $W_{\mathbf{j}}$ with length parameter $\lfloor\theta_C^3\rfloor$ and define C-traversable, -super, and -evil by replacing the B-vacancies with C-vacancies. Recall that we assume $q_A\theta^3_{q_C}/\log_2(\theta_B)\to\infty$ as $q_B\to 0$ so that the results follow analogously for C-vacancies with minor adjustments. We omit details here that lead to the result that

$$\mu(\mathbb{1}_{\mathcal{E}^{(C,*)}}\mathrm{Var}^{(AC)}_{x_{\mathbf{j}_C}}(f))\leq 2^{\theta_B^2(1+\varepsilon)/4}\mu\Big[\mathcal{D}_{\mathcal{Q}^{(C,*)}}(f)\Big],\tag{9.6}$$

for $\mathbf{j}_C=-3\mathbf{e}_1+\mathbf{e}_2$ and q_B small enough.

As in the low vacancy density case we need a final event that brings the B- resp. C-vacancy from $\mathcal{Q}^{(h,*)}(x)$ to x. To that end, let us define some paths (see Figure 9.7).

- Let $\Gamma^{(B)}$ be a shortest path starting at $\mathbf{e}_2$ and ending at the first vertex neighbouring $EW_{\mathbf{j}_B}\setminus W_{\mathbf{j}_B}$ that first goes straight up and then right.
- Let $\Gamma^{(B,left)}$ be the path that starts at $-\lfloor\theta_C^3\rfloor\mathbf{e}_1+\mathbf{e}_2$ and is straight until $-\mathbf{e}_1+\mathbf{e}_2$ and then equal to $(\Gamma^{(B)}-\mathbf{e}_1)\setminus\Gamma^{(B)}$. Let $\Gamma^{(B,right)}$ be the path starting at $\mathbf{e}_1+\lfloor\theta_B^3\rfloor\mathbf{e}_2$ that goes straight up until it hits $\Gamma^{(B)}-\mathbf{e}_2$, which it then follows to the right.
- Let $\Gamma^{(C)}$ be the shortest path that starts at $-\mathbf{e}_1$ and goes straight left and then up that ends up at a vertex neighbouring $EW_{\mathbf{j}_C}\setminus W_{\mathbf{j}_C}$. Let $x^{(C)}$ be the point where the path switches from going left to going up.
- Let $\Gamma^{(C,left)}$ be the union of $\Gamma^{(C)}-\mathbf{e}_1-\mathbf{e}_2$ and $\{x^{(C)}-\lfloor\theta_C^3\rfloor\mathbf{e}_1,\dots,x^{(C)}-\mathbf{e}_1\}$.

Notice that since $\Gamma^{(B,left)}$ starts at $-\lfloor\theta_C^3\rfloor\mathbf{e}_1+\mathbf{e}_2$ the various paths do not intersect. We define $\mathcal{E}^{(0)}$ as the $\omega\in\Omega$ such that

- $\omega_x\in\{\star,A,B\}$ for any $x\in\Gamma^{(B)}$.
- $\omega_x\in\{\star,A\}$ for any $x\in\Gamma^{(B,left)}\cup\Gamma^{(B,right)}$ and there is at least one A-vacancy on $\Gamma^{(B,left)}\setminus\{\Gamma^{(B)}-\mathbf{e}_1\}$ and on $\Gamma^{(B,right)}\setminus\{\Gamma^{(B)}-\mathbf{e}_2\}$.
- $\omega_x\in\{\star,A,C\}$ for any $x\in\Gamma^{(C)}$.
- $\omega_x\in\{\star,A\}$ for any $x\in\Gamma^{(C,left)}$ at least one A-vacancy on $\Gamma^{(C,left)}\setminus(\Gamma^{(C)}-\mathbf{e}_1-\mathbf{e}_2)$.

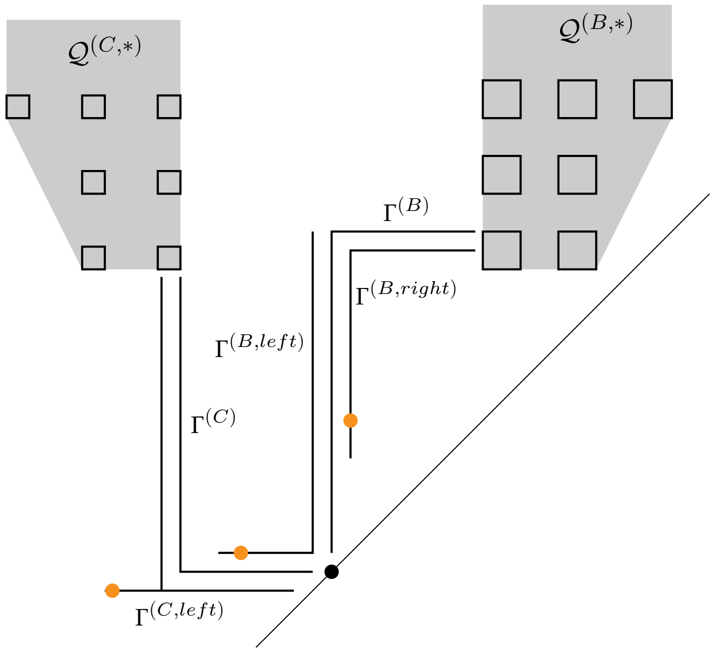

FIGURE 9.7 Image of the various Γ paths and the exemplary A-vacancies where $\mathcal{E}^{(0)}$ requires them. Note the different sizes of $\mathcal{Q}^{(B,*)}$ and $\mathcal{Q}^{(C,*)}$.

The support of $\mathcal{E}^{(0)}$ by construction has no intersection with $\mathcal{Q}^{(B,*)}$ and $\mathcal{Q}^{(C,*)}$. Let $\mathcal{E}^{(*)} := \mathcal{E}^{(0)} \cap \mathcal{E}^{(B,*)} \cap \mathcal{E}^{(C,*)}$ and let $\mathcal{E}_x^{(*)}$ be the translated version by $x \in \mathbb{Z}^2$. Then the family $\{\mathcal{E}_x^{(*)}\}_{x\in\mathbb{Z}^2}$ satisfies the exterior condition w.r.t. to the same family of sets as given in Lemma 9.11. Using the assumptions on $\mathbf{q}$ it is straightforward to check that

$$\lim_{q_B\to 0} \operatorname{Supp}(\mathcal{E}^{(0)})\mu(1-\mathcal{E}^{(0)}) = 0.$$

Combining this with Lemma 9.20 and the results from Section 9.1.2 we can apply the exterior condition theorem, Theorem 8.2. Further, $\mathcal{E}^{(0)}$ fulfills that analogous role to the eponymous event in Section 9.2 as we see in the next Lemma.

Lemma 9.27. *Let $\mathcal{A}$ be the event defined by the intersection*

$$\mathcal{A} := \mathcal{E}^{(0)} \cap \{EW_{\mathbf{j}_B}\ B\text{-}super\} \cap \{EW_{\mathbf{j}_C}\ C\text{-}super\}.$$

Then,

$$\mu(\mathbb{1}_{\mathcal{A}}\operatorname{Var}_0(f)) \leq 2^{\kappa\theta_B \log_2(\theta_B)}\mathcal{D}_{\operatorname{Supp}(\mathcal{A})}(f).$$

for q_B small enough.

Sketch of the proof. We only give a sketch since the employed techniques are always the same. Extending the variance, Lemma 9.15 and using block relaxation, Lemma 6.17, gives

$$\mu(\mathbb{1}_{\mathcal{A}}\operatorname{Var}_0(f)) \leq \frac{C}{q_B}\mu\Big[\mathbb{1}_{\mathcal{A}}\Big(\operatorname{Var}_{\mathbf{e}_2}^{(AB)}(f) + \mathbb{1}_{\omega_{\mathbf{e}_2}=B}\operatorname{Var}_0(f)\Big)\Big].$$

For the first summand, given $\mathcal{A}$, we can use a combination of the block relaxation Lemma (Lemma 6.17) and Lemma 9.24 to get an appropriate upper bound. For the second summand we can repeat the calculation for the C-vacancy side to get a term

$$\frac{C}{q_A q_B q_C}\mu(\mathbb{1}_{\mathcal{A},\omega_{\mathbf{e}_2}=B,\omega_{-2\mathbf{e}_1}=C,\omega_{-2\mathbf{e}_1-\mathbf{e}_2}=A}\mathrm{Var}_0(f)).$$

Write $\mathrm{Var}_0(f)$ as a sum of transition terms using Lemma 8.5, for the B-transition use that $\mathbb{1}_{\omega_{\mathbf{e}_2}=B} \le c_0^B$ and for the A and C transition terms we can use the path method recalling that for $\omega \in \mathcal{A}$ we have $\omega_{-\mathbf{e}_1} \in \{\star, A, C\}$ (analogously to, for example, the situation in Figure 9.6). The claim follows. □

Now we can finally prove that in the case of $q_A = q_{\max}$ we get the correct upper bound on the relaxation time.

Proposition 9.28. *For parameter sets as fixed in the beginning of the section with* $q_{\max} = q_A$ *we have*

$$\lim_{q_B \to 0} \frac{\gamma(G, \mathbf{q})}{\gamma_2(q_B)} \ge 1.$$

Sketch of proof. We can use the exterior condition theorem to get

$$\mathrm{Var}(f) \le 4 \sum_{x \in \mathbb{Z}^2} \mu(\mathbb{1}_{\mathcal{E}_x^{(*)}} \mathrm{Var}_x(f))$$

for q_B small enough. Let us bound a generic summand and w.l.o.g. consider $x = 0$. We can extend the variance and use the block relaxation Lemma (twice) to get

$$\mu(\mathbb{1}_{\mathcal{E}^{(*)}}\mathrm{Var}_0(f)) \le 2^{\kappa\theta_B}\mu(\mathbb{1}_{\mathcal{A}}\mathrm{Var}_0(f) + \mathbb{1}_{\mathcal{E}^{(B,*)}}\mathrm{Var}^{(AB)}_{x_{\mathbf{j}_B}}(f) + \mathbb{1}_{\mathcal{E}^{(C,*)}}\mathrm{Var}^{(AC)}_{x_{\mathbf{j}_C}}(f))$$

for q_B small enough with $\mathcal{A}$ from Lemma 9.27. The claim follows by using Lemma 9.27 for the first summand, Equation (9.5) for the second summand and Equation (9.6) for the third summand and the fact that the intersection between $\mathrm{Supp}(\mathcal{E}_x^{(*)})$ for the various $x \in \mathbb{Z}^2$ in the sum over x is of the order $2^{O(\theta_B)}$. □

We never explicitly used that $q_C > q_B$ so the same result also holds in the case $q_B = q_{\mathrm{med}}$ and $q_C = q_{\min}$. Further, by symmetry this also covers the case $q_B = q_{\max}$. □

9.3.2 The case $q_{\max} = q_C$

Recall that the single frequent vacancy type case, part (3.ii) of Theorem 6, deals with parameter sets such that $q_{\max}\theta^3_{q_{\mathrm{med}}} / \log_2(\theta_{q_{\min}}) \to \infty$ and $q_{\mathrm{med}}\theta^6_{q_{\min}} \to 0$ as $q_{\min} \to 0$. By symmetry Section 9.3.1 covers the cases when one of the non-central vacancies has the maximum equilibrium so we are left with the case when the central vacancy, i.e. the C-vacancy, has the highest equilibrium density. The proof is, in fact, analogous and we will just discuss how to adapt the enlarged boxes $EW_{\mathbf{j}}$. How to adapt the analogous results based on these new definitions should be evident from that point on.

Definition 9.29 (h-traversable)**.** Let $L = \lfloor \theta_h^3 \rfloor$, let $\Lambda_{\mathbf{j}}$ be the equilateral box of side length $L-1$ and origin at $(L+1)\mathbf{j}$ and let $W_{\mathbf{j}}$ be its outline, i.e. the shortest cycle containing $(L+1)\mathbf{j} + \{0, (L-1)\mathbf{e}_1, (L-1)\mathbf{e}_2, (L-1)(\mathbf{e}_1+\mathbf{e}_2)\}$. Further let $EW_{\mathbf{j}} = \tilde{\Lambda} \setminus \Lambda_{\mathbf{j}}$ where $\tilde{\Lambda}$ is the equilateral box of side length L such that the *top-left corner* of $\tilde{\Lambda}$ and $\Lambda_{\mathbf{j}}$ coincide (instead of the origin as in the previous section). Write $x_{\mathbf{j}}$ for the *bottom-right corner* of $EW_{\mathbf{j}}$. For $h \neq C$ we call $EW_{\mathbf{j}}$

- *h-traversable* if

- $\omega_x \in \{\star, C\}$ for any $x \in W_{\mathbf{j}}$,
- $\omega_x \in \{\star, C, h\}$ for any $x \in EW_{\mathbf{j}} \setminus W_{\mathbf{j}}$ and
- for any $i \in [2]$ there is at least one C-vacancy on the top and left boundary of $W_{\mathbf{j}}$.

- h-*super* if $EW_{\mathbf{j}}$ is h-traversable and $\omega_{x_{\mathbf{j}}} = h$.
- h-*evil* if it is not h-traversable.

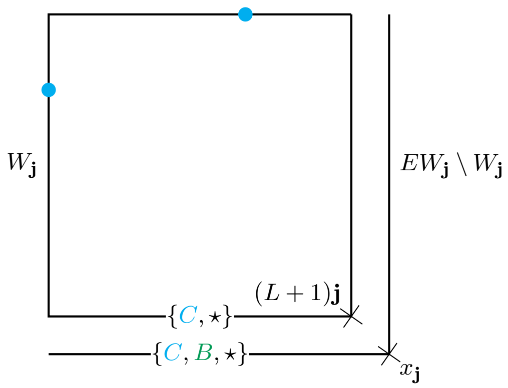

FIGURE 9.8 Illustration of a B-traversable $EW_{\mathbf{j}}$ in the case $q_{\max} = q_C$. Note that the C-vacancies on the top and left boundary can be anywhere on that boundary.

Now $EW_{\mathbf{j}}$ is h-traversable if the bottom and right boundaries of $EW_{\mathbf{j}}$ are in the state $\{\star, C, h\}$ since C-vacancies move down and right, so we need to find the C-vacancies on the left and top boundary to free the path for any h-vacancy on $EW_{\mathbf{j}} \setminus W_{\mathbf{j}}$. Apart from changing these small details in the constructions the rest is equivalent so that Theorem 6(3.ii) follows together with the result from the last section. $\square$

9.4 Single low density vacancy type: Proof of Theorem 6(3.iii)

In this section consider again the G-MCEM with $G = \{A, B, C\}$ this time with a parameter set $\mathbf{q}$ such that $q_{\min} = q_B$ and $\liminf_{q_B \to 0} q_{\text{med}} > 0$, i.e. there is a constant $\lambda > 0$ with $q_A, q_C > \lambda$ for q_B small enough. This covers case (3.iii) since both A- and C-vacancies share the direction $\mathbf{e}_1$, the other case in which $q_{\min} = q_A$, is equivalent to the present case by symmetry.

Using that both A- and C-vacancies have a high equilibrium density, we find configurations that clear any non-B-vacancy in the $\mathbf{e}_1$-direction. As in previous proofs we work with block lattices. In this section we let $\{W_{\mathbf{j}}\}_{\mathbf{j} \in \mathbb{Z}^2}$ be the block lattice given by boxes

$$W_{\mathbf{j}} = (\mathbf{j}_1, 3\mathbf{j}_2) + \{0, \mathbf{e}_2, 2\mathbf{e}_2\}.$$

We call $(\mathbf{j}_1, 3\mathbf{j}_2)$ the *lower vertex of* $W_{\mathbf{j}}$, $(\mathbf{j}_1, 3\mathbf{j}_2 + 2)$ the *upper vertex*, the set of lower and upper vertices we then call the *outer vertices* and $(\mathbf{j}_1, 3\mathbf{j}_2 + 1)$ the *central vertex*. The associated local state space is $\Omega^*_{\mathbf{j}} := \{0,1\}^{W_{\mathbf{j}}}$, the equilibrium measure is $\mu^*_{\mathbf{j}} = \mu^*_{W_{\mathbf{j}}}$ and the variance is $\operatorname{Var}^*_{\mathbf{j}}(f) = \operatorname{Var}_{W_{\mathbf{j}}}(f)$. For $\omega \in \Omega^*$ we say that $W_{\mathbf{j}}$

- is B-traversable, if there is no B on the outer vertices.
- is B-super, if it is B-traversable and the central vertex is B.

- is AC-traversable, if there is no B on $W_{\mathbf{j}}$.
- is AC-super, if it is AC-traversable, the lower vertex is A and the upper vertex is C.

Remark 9.30. To justify the above definitions and the recycling of the traversable and super names let us give a high level overview of what we do with these states to prepare the reader for the detailed calculations. Recall that A-vacancies propagate north and east, while C-vacancies propagate south and east. In an AC-super box the central vertex is always facilitated for any transition from A or C to the neutral state and vice versa. By A- and C-vacancies sharing the east propagation direction this extends to any vertex in an AC-traversable box to the east of an AC-super box (see Lemma 9.32).

Further, if there are any B-traversable or B-super boxes to the East of an AC-super box, following at least one AC-traversable box, we can also remove any non B-vacancy from the central vertex. This is what allows us in Lemmas 9.33 and 9.35 to propagate the B-vertices from B-super boxes on paths of B-traversable boxes given an appropriate configuration of AC-super and traversable boxes.

As in the previous proofs our goal is to define a set of events $\{\mathcal{E}_{\mathbf{j}}\}_{\mathbf{j}\in\mathbb{Z}^2}$ on which we can use the exterior condition theorem and where $\mathcal{E}_x$ allows us to recover a Dirichlet form of the ABC-model starting from a term like $\mu(\mathbb{1}_{\mathcal{E}_{\mathbf{j}}}\mathrm{Var}_{W_{\mathbf{j}}}(f))$ at a cost $2^{\theta_B^2(1+\varepsilon)/4}$ for q_B small enough.

For this we cannot use $\mathcal{Q}^{(B)}$ anymore since there is no obvious relaxation scheme that allows us to transport B-vacancies on coarse-grained B-paths (as in the proof of part (3.ii)). Since the A-vacancies have a high frequency we also do not have to make a construction that stays above the diagonal as in Lemma 9.11 to satisfy the exterior condition. We can work with the set V_0 given by the vertices 'below' the line that goes through the origin and $2^{\theta_B^2}\mathbf{e_1}$ and define $V_n = V_0 + n\mathbf{e}_2$ for any $n \in \mathbb{Z}^2$ so that $\{V_n\}_{n\in\mathbb{N}}$ is an increasing and exhausting set. This allows us to construct a lattice of straight lines of side length at most $2^{\theta_B^2}$ in the positive quadrant and still put a condition on the line going in the $-\mathbf{e}_1$ direction while satisfying the exterior condition.

Let $\ell = \lceil\theta_B^{3/2}\rceil$ and $N = 2^{\lceil\theta_{q_B}/2+\log_2(\theta_B)\rceil}$. For $i \in [N]$ we call the box of side lengths $(N\ell-1, N-1)$ with origin at $i\ell\mathbf{e}_1 + \mathbf{e}_2$ the i-th vertical strip $Q_i^{(v)}$. For $j \in [0, N]$ we call the box with side lengths $(N-1, N\ell-1)$ and origin at $(j\ell+1)\mathbf{e}_2$ the j-th horizontal strip $Q_j^{(h)}$. We denote by $Q_{i,j}$ the equilateral box of side length $\ell-1$ given by $Q_i^{(v)} \cap Q_j^{(h)}$. The union $\mathcal{Q}^{(B)}$ of all strips is an equilateral box of side length $N\ell-1$ and origin $\mathbf{e}_2$. In particular, note that any family of events $\{\mathcal{E}_x\}_{x\in\mathbb{Z}^2}$ with $\mathcal{E}_x$ supported on $\mathcal{Q}^{(B)} + x$ satisfies the exterior condition w.r.t. the above family $\{V_n\}_{n\in\mathbb{Z}}$.

The dynamics to propagate B-vacancies on horizontal and vertical paths is different. A- and C-vacancies only share the $\mathbf{e}_1$ direction so that AC-super boxes can only propagate in an $\mathbf{e}_1$ directions, which means that for each row we want to move a B-super box vertically, we need an AC-super box somewhere that removes any A- or C-vacancies. To propagate B-super boxes horizontally a single AC-super box suffices. Thus vertically we need boxes that guarantee us the AC-super vertices.

Definition 9.31 (Vertical crossing)**.** Consider a box $\Lambda \subset Q_i^{(v)}$ of side lengths $(\lfloor\theta_B^{5/4}\rfloor - 1, N-1)$ with origin $\mathbf{j}_0$. Let $\partial^{(r)}\Lambda$ be the right boundary of Λ, i.e. the $\mathbf{j} \in \Lambda$ such that $\mathbf{j}\cdot\mathbf{e}_1 = \mathbf{j}_0\cdot\mathbf{e}_1 + \lfloor\theta_B^{5/4}\rfloor - 1$. For $\omega \in \Omega^*_{\mathcal{Q}^{(B)}}$, Λ is a *vertical crossing of* $Q_i^{(v)}$ if

- $W_{\mathbf{j}}$ is B-traversable for any $\mathbf{j} \in \partial^{(r)}\Lambda$.
- $W_{\mathbf{j}}$ is AC-traversable for any $\mathbf{j} \in \Lambda \setminus \partial^{(r)}\Lambda$.
- There is at least one $\mathbf{j}$ per row in $\Lambda \setminus (\partial^{(r)}\Lambda \cup (\partial^{(r)}\Lambda - \mathbf{e}_2))$ such that $W_{\mathbf{j}}$ is AC-super.

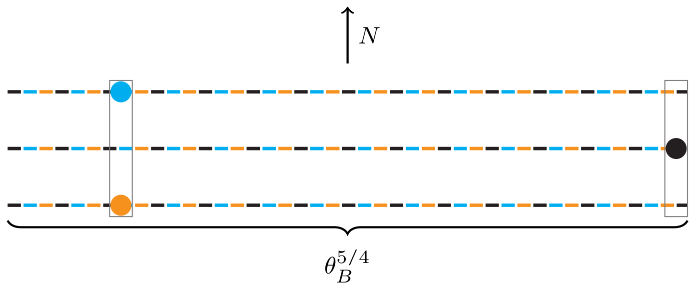

FIGURE 9.9 One of the N horizontal sections of a vertical crossing (see Definition 9.31). Three vertically aligned vertices (e.g. the rectangles drawn) are one box $W_{\mathbf{j}}$. The right box is on the right boundary and thus by assumption B-traversable, so that there is no condition on the central vertex (black dot). All other vertices have no B-vacancy, indicated by the striped path, orange for A-vacancies, blue for C and black for neutral state. The left box is the AC-super box implied by the definition of vertical crossings.

The main idea behind this definition is the following: To propagate a B-super box on the right boundary, we use that on each row there is an AC-super box on a line of AC-traversable boxes. This AC-super box can remove any A- or C-vacancy from the AC-traversable part and then in particular also from the B-traversable part on the right boundary of Λ, which then allows the B-vacancy in the B-super box to move down. Let us isolate this horizontal motion of AC-super boxes. Recall for this, that we write $\mu^{(AC)}$ and $\mathrm{Var}^{(AC)}$ to denote the measure resp. variance conditioned on there only being A and C vacancies and that by definition

$$\mathrm{Var}^*_{\mathbf{j}}(f \mid AC\text{-traversable}) = \mathrm{Var}^{(AC)}_{W_{\mathbf{j}}}(f).$$

Lemma 9.32. *We find a constant such that*

$$\mu(\mathbb{1}_{W_{-\mathbf{e}_1}\ AC\text{-super}}\mathrm{Var}^*_{\mathbf{0}}(f \mid AC\text{-traversable}) \leq C\mu(\mathcal{D}_{W_{-\mathbf{e}_1}\cup W_{\mathbf{0}}}(f)).$$

Proof. We can write the variance as transition terms (Lemma 8.5)

$$\begin{aligned}\mu(\mathbb{1}_{W_{-\mathbf{e}_1}\ AC\text{-super}}\mathrm{Var}^{(AC)}_{W_{\mathbf{0}}}(f)) &\leq \sum_{x\in W_{\mathbf{0}}}\mu(\mathbb{1}_{W_{-\mathbf{e}_1}\ AC\text{-super}}\mathrm{Var}^{(AC)}_{x}(f)\\ &\leq C\sum_{x\in W_{\mathbf{0}}}\sum_{h\in\{A,C\}}\mu(\mathbb{1}_{W_{-\mathbf{e}_1}\ AC\text{-super}}pq_h(\nabla^{(h)}_x(f))^2.\end{aligned}$$

Let ω be such that $W_{-\mathbf{e}_1}$ is AC-super and $W_{\mathbf{0}}$ AC-traversable (see Figure 9.10 for an example with the legal path we now construct). Consider the case $x = 2\mathbf{e}_2$ and some $h \in \{A, C\}$. Since the upper vertex of $W_{-\mathbf{e}_1}$ is C and the lower vertex is A, the central vertex of $W_{-\mathbf{e}_1}$ can legally transition to $\star$ and then in particular to any state. Since $\mathbf{e}_1 \in \mathcal{P}(h)$ then also the central vertex of $W_{\mathbf{0}}$ can transition to any state in a legal path of length $O(1)$. Thus there is a legal path putting any state on $2\mathbf{e}_2$. Indeed, by assumption $\omega_{2\mathbf{e}_2-\mathbf{e}_1} = C$ and we just showed that we can put $\omega_{\mathbf{e}_2} = A$.

We can conclude with the path method since this argument applies analogously to the lower vertex of $W_{\mathbf{0}}$ and the cost of changing measures is a term depending on q_A and q_C which we can estimate by a constant by assumption on $\mathbf{q}$. $\square$

With this we can show how B-super boxes propagate vertically on vertical crossings.

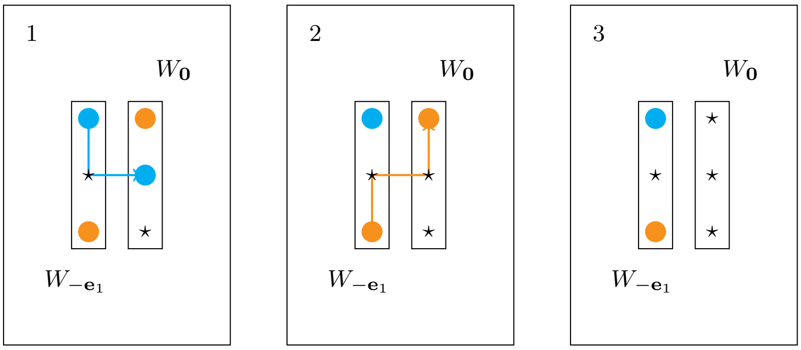

FIGURE 9.10 Example of the sequence of transitions in a legal path that ends in a configuration in which $W_{\mathbf{0}}$ is in a neutral state using that $W_{-\mathbf{e}_1}$ is AC-super.

Lemma 9.33 (Vertical propagation)**.** *Let $\Lambda \subset Q_i^{(v)}$ as in Definition 9.31, and let $\mathbf{j}^{(1)} \in \partial^{(r)}\Lambda \cap Q_{i,0}$. Let $\mathcal{A}^{(v)}$ be the event given by the ω such that Λ is a vertical crossing of $Q_i^{(v)}$ and there is a $\mathbf{j}^{(2)} \in \partial^{(r)}\Lambda \cap Q_{i,1}$ such that $W_{\mathbf{j}^{(2)}}$ is B-super. Then,*

$$\mu\Big(\mathbb{1}_{\mathcal{A}^{(v)}}\mathrm{Var}^*_{\mathbf{j}^{(1)}}(f \,|\, \mathcal{A}^{(v)})\Big) \le 2^{\kappa\theta_B \log_2(\theta_B)}\mu(\mathcal{D}_{Q_{i,0}\cup Q_{i,1}}(f)).$$

Proof. W.l.o.g. assume that the right boundary $\partial^{(r)}\Lambda$ of Λ is on the vertical axis such that $\mathbf{j}^{(1)} = \mathbf{0}$ and assume also w.l.o.g. that the B-super $\mathbf{j}^{(2)}$ implied by $\mathcal{A}^{(v)}$ is on the furthest vertex in $\partial^{(r)}\Lambda \cap Q_{i,1}$ from the origin, i.e. $\mathbf{j}^{(2)} = (2\ell - 1)\mathbf{e}_2$. Let $\Gamma = \{\mathbf{0}, \mathbf{e}_2, \ldots, \mathbf{j}^{(2)} - \mathbf{e}_2\}$ be the part of the right boundary starting at $\mathbf{j}^{(1)}$ and stopping right before $\mathbf{j}^{(2)}$. Let $c_{\mathbf{j}}^{(v)}$ be the constraint given by the indicator over the event that $W_{\mathbf{j}+\mathbf{e}_2}$ is B-super if $\mathbf{j} \neq \mathbf{j}^{(2)} - \mathbf{e}_2$ and 1 if $\mathbf{j} = \mathbf{j}^{(2)} - \mathbf{e}_2$.

Consider the auxiliary process on Γ with the constraints $c_{\mathbf{j}}^{(v)}$ that, if $W_{\mathbf{j}}$ is unconstrained, samples from all B-traversable states on $W_{\mathbf{j}}$. The equilibrium measure of this process is given by $\mu_\Gamma^{(*,BT)} := \otimes_{\mathbf{j}\in\Gamma}\mu^*_{\mathbf{j}}(\cdot \,|\, B\text{-traversable})$. Since $\mu_{\mathbf{j}}^{(*,BT)}(B\text{-super}) = \mu^*_{\mathbf{j}}(B\text{-super} \,|\, B\text{-traversable}) = q_B$ the spectral gap of this process is equal to the spectral gap of the one-dimensional East model with vacancy density q_B on Γ with good boundary conditions.

Hence, we can extend the variance (Lemma 9.15) and use Theorem 2.3 to get

$$\mu\Big(\mathbb{1}_{\mathcal{A}^{(v)}}\mathrm{Var}^*_{\mathbf{j}^{(1)}}(f \,|\, \mathcal{A}^{(v)})\Big) \le \mu\Big(\mathbb{1}_{\mathcal{A}^{(v)}}\mathrm{Var}_{\mu_\Gamma^{(*,BT)}}(f)\Big) \le 2^{\kappa\theta_B \log_2(\theta_B)}\sum_{\mathbf{j}\in\Gamma}\mu\Big(\mathbb{1}_{\mathcal{A}^{(v)}}c_{\mathbf{j}}^{(v)}\mathrm{Var}_{\mu_{\mathbf{j}}^{(*,BT)}}(f)\Big).$$

Consider the summand for $\mathbf{j} = \mathbf{0}$ and let $\omega \in \mathcal{A}^{(v)}$. Let $V^{(0)} = \cup_{j=0}^3 j\mathbf{e}_2$ be the union of vertices in $W_{\mathbf{0}}$ together with the lower vertex of $W_{\mathbf{e}_2}$ and recall that by $c_{\mathbf{0}}^{(v)}$ the vertex $4\mathbf{e}_2$ has a B-vacancy. Further, let $V^{(i)} = \cup_{j=\{0,1\}}W_{-i\mathbf{e}_1+j\mathbf{e}_2}$ for $i \in [2]$ and let $V = V^{(0)} \cup V^{(2)}$. Recall that by $\mathcal{A}^{(v)}$ the boxes in $V^{(i)}$ are AC-traversable and define further $\tilde{\mathcal{A}}$ as the event that $W_{\mathbf{j}}$ is AC-super for $\mathbf{j} \in V^{(2)}$. We can extend the variance to V and use the block relaxation Lemma (Lemma 6.17) to get

$$\begin{aligned}\mu\Big(\mathbb{1}_{\mathcal{A}^{(v)}}c_{\mathbf{0}}^{(v)}\mathrm{Var}_{\mu_{\mathbf{0}}^{(*,BT)}}(f)\Big) &\le \mu\Big(\mathbb{1}_{\mathcal{A}^{(v)}}c_{\mathbf{0}}^{(v)}\mathrm{Var}_V(f \,|\, \mathcal{A}^{(v)})\Big)\\ &\le C\mu\Big[\mathbb{1}_{\mathcal{A}^{(v)}}c_{\mathbf{0}}^{(v)}\Big(\mathbb{1}_{\tilde{\mathcal{A}}}\mathrm{Var}_{V^{(0)}}(f \,|\, \mathcal{A}^{(v)}) + \mathrm{Var}_{V^{(2)}}(f \,|\, \mathcal{A}^{(v)})\Big)\Big]. \qquad (9.7)\end{aligned}$$

We upper bound the two summands separately. For the first term we get

$$\mu\Big(\mathbb{1}_{\mathcal{A}^{(v)},\tilde{\mathcal{A}}}c_{\mathbf{0}}^{(v)}\mathrm{Var}_{V^{(0)}}(f \,|\, \mathcal{A}^{(v)})\Big) \le 2^{\kappa\theta_B}\mu\Big(\mathcal{D}_{\cup_{i\in[0,2]}V^{(j)}}(f)\Big). \qquad (9.8)$$

This is done through the path method analogous to Lemma 9.32. Indeed, by $\mathcal{A}^{(v)}$ we know that the boxes in $V^{(1)}$ are AC-traversable and by $\tilde{A}$ that the boxes in $V^{(2)}$ are AC-super. Start by writing the variance as a sum of transition terms. Any transition on $V^{(1)}$ from h to $\star$ or vice versa for $h \in \{A, C\}$ can be done at a constant cost since we can, following the proof of Lemma 9.32, put any $\{A, C, \star\}$ state on $V^{(1)}$. This also gives us a legal path for transitions from $\star$ to a vacancy type in $\{A, C\}$ on $V^{(0)}$, since $\mathbf{e}_1$ is a propagation direction of both A- and C-vacancies. For the B-vacancy transition on the central vertex of $W_{\mathbf{0}}$ we can use the same legal path to put $\star$ between central vertex of $W_{\mathbf{0}}$ and the one of $W_{\mathbf{e}_2}$ and finally move the B-vacancy of the B-super $W_{\mathbf{e}_2}$ to the central vertex of $W_{\mathbf{0}}$. Equation (9.8) then follows from the path method where the main cost comes from the change of measure giving the term $2^{\kappa q_B}$.

For the second summand in Equation (9.7) first split up the variance

$$\mathrm{Var}_{V^{(2)}}(f \mid \mathcal{A}^{(v)}) \leq \mu_{V^{(2)}}(\mathrm{Var}^*_{-2\mathbf{e}_2}(f \mid \mathcal{A}^{(v)}) + \mathrm{Var}^*_{-\mathbf{e}_2+\mathbf{e}_1}(f \mid \mathcal{A}^{(v)}) \mid \mathcal{A}^{(v)})$$

and consider the variance over $W_{-2\mathbf{e}_2}$. The upper bound for the second term follows analogously.

Consider an auxiliary process with the constraints $c^{(h)}_{\mathbf{j}}$ given by the indicator over the event that $W_{\mathbf{j}-\mathbf{e}_1}$ is AC-super. If $W_{\mathbf{j}}$ is unconstrained in this process, sample it from all AC-traversable states at a legal ring. This process has the same spectral gap as the East model with vacancy density $\frac{q_A q_C}{(q_A+q_C)^2}$. Using that $\mathcal{A}^{(v)}$ implies that there is an AC-super box to the left of $W_{\mathbf{e}_2}$ we can use the enlargement trick ([13]*Lemma 3.6, which immediately generalises to this case), to get

$$\begin{aligned}
\mu\Big(\mathbb{1}_{\mathcal{A}^{(v)}}\mathrm{Var}^*_{-2\mathbf{e}_2}(f \,|\, \mathcal{A}^{(v)})\Big) &\leq C \sum_{j=2}^{\lfloor \theta_B^{5/4}\rfloor-2} \mu(c^{(h)}_{-j\mathbf{e}_1}\mathrm{Var}^*_{-j\mathbf{e}_1}(f \,|\, AC\text{-traversable})) \\
&\leq C \sum_{j=2}^{\lfloor \theta_B^{5/4}\rfloor-2} \mu(\mathcal{D}_{W_{-j\mathbf{e}_1}\cup W_{-(j+1)\mathbf{e}_1}}(f))
\end{aligned}$$

where in the second inequality we used Lemma 9.32. Combining the two estimates gives the claim after taking into account that the vertices in $V^{(2)}$ are counted twice which we absorb into κ. □

Remark 9.34. Notice that here we lose the indicator over $\mathcal{A}^{(v)}$ since it requires there to be no B-vacancy between the central vertices but the path method adds these transitions. This will be important later, as keeping the indicators was important for taking the sum over the possible grids $\mathcal{C}$.

The horizontal paths will consist of B-traversable $W_{\mathbf{j}}$ that connect the vertical crossings. We isolate here the result that allows us to propagate a central B on these horizontal paths.

The basic situation is as follows. Let $\Gamma = \Gamma^{(l)} \cup \Gamma^{(r)}$ with $\Gamma^{(l)} = [-\lfloor \theta_B^{5/4} \rfloor \mathbf{e}_1, \dots, -\mathbf{e}_1]$ and $\Gamma^{(r)} = [\mathbf{0}, \dots, \ell\mathbf{e}_1 - 1]$. Let $\mathcal{A}^{(h)}$ be the event that $w_{\mathbf{j}}$ for $\mathbf{j} \in \Gamma^{(l)}$ is AC-traversable, that there is an AC-super $W_{-i\mathbf{e}_1}$ for $i \leq -3$, that $W_{\mathbf{j}}$ for $\mathbf{j} \in \Gamma^{(r)}$ is B-traversable and that $W_{\ell\mathbf{e}_1}$ is B-super (see Figure 9.11).

Lemma 9.35 (Horizontal propagation)**.** *For Γ and $\mathcal{A}^{(h)}$ as above we find a constant κ such that*

$$\mu(\mathbb{1}_{\mathcal{A}^{(h)}}\mathrm{Var}^*_{\mathbf{0}}(f \,|\, \mathcal{A}^{(h)})) \leq 2^{\kappa\theta_B^{3/2}}\mu(\mathcal{D}_{W(\Gamma)}(f)),$$

where $W(\Gamma) = \cup_{\mathbf{j}\in\Gamma} W_{\mathbf{j}}$

Proof. Split $W(\Gamma^{(r)})$ into $W^{(ro)} \cup W^{(rc)}$ of respectively the set of outer and central vertices. Define the event $\tilde{A}$ that there are only C-vacancies on the upper vertices of $W^{(ro)}$ and only A-vacancies the lower vertices. Then, we can extend the variance (Lemma 9.15) and use the block relaxation Lemma (Lemma 6.17) to find a constant κ such that

$$\mu(\mathbb{1}_{\mathcal{A}^{(h)}}\mathrm{Var}^*_{\mathbf{0}}(f \mid \mathcal{A}^{(h)})) \leq \mu(\mathbb{1}_{\mathcal{A}^{(h)}}\mathrm{Var}^*_{\Gamma^{(r)}}(f \mid \mathcal{A}^{(h)}))$$

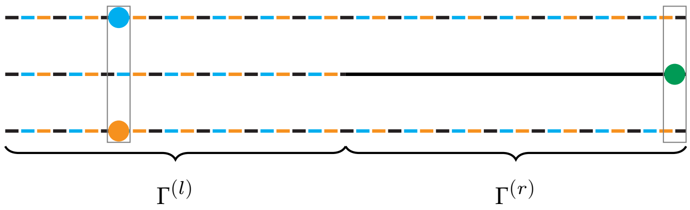

FIGURE 9.11 Path Γ in the context of the horizontal propagation Lemma 9.35. The right box is the B-super box and the left box the AC-super box. The black path indicates that the boxes in $\Gamma^{(r)}$ are B-traversable so that there is no condition on the central vertices.

$$\leq 2^{\kappa\theta_B^{3/2}}\mu\Big[\mathbb{1}_{\mathcal{A}^{(h)}}\Big(\mathbb{1}_{\tilde{A}}\mathrm{Var}_{W^{(rc)}}(f)+\mathrm{Var}^{(AC)}_{W^{(ro)}}(f)\Big)\Big]. \tag{9.9}$$

Consider the first summand. On $\mathcal{A}^{(h)}$ there is a B-vacancy to the right of $W^{(rc)}$, so consider the auxiliary model with the standard B-vacancy constraints c_x^B that samples from μ_x at a legal ring on $x \in W^{(rc)}$. Given $\mathbb{1}_{\mathcal{A}^{(h)}}$ this auxiliary model has good boundary conditions and the same spectral gap as the East model with vacancy density q_B so that

$$\mu(\mathbb{1}_{\mathcal{A}^{(h)},\tilde{A}}\mathrm{Var}_{W^{(rc)}}(f)) \leq 2^{\kappa\theta_B\log_2(\theta_B)}\sum_{x\in W^{(rc)}}\mu(\mathbb{1}_{\tilde{A}}c_x^B\mathrm{Var}_x(f)).$$

Now write the variances as transition terms using Lemma 8.5 and use that with $\tilde{A}$ and c_x^B every $x \in W^{(rc)}$ is unconstrained for every transition so that

$$\sum_{x\in W^{(rc)}}\mu(\mathbb{1}_{\tilde{A}}c_x^B\mathrm{Var}_x(f)) \leq C\mu(\mathcal{D}_{W^{(rc)}}(f)).$$

For the second summand in Equation (9.9) write $\mathrm{Var}^{(AC)}_{W^{(ro)}}(f)$ as a sum of transition terms for A- and C-vacancy transitions. We saw in Lemma 9.32 how an AC-super state can put any state on an AC-traversable state to its right. Given an AC-super and then an AC-traversable state we can thus put any state in $\{\star, A, C\}$ onto the upper or lower vertices of boxes right to them, if they don't contain B-vacancies. The legal path dynamic is completely analogous to the one in Lemma 9.32 so we omit the details. The lengths of the paths are $O(|W^{(ru)}|) = O(\theta_B^{3/2})$, so the path method gives an upper bound of the order $2^{\kappa\theta_B^{3/2}}$ and the claim follows. □

We now come to the grids we use in this section (see Figure 9.12).

Definition 9.36 (Grid)**.** Call a union of $\mathcal{C} = \cup_{i\in[N]}\mathcal{C}_i^{(h)} \cup \mathcal{C}_j^{(v)}$ a *grid* if $\mathcal{C}_i^{(h)} \subset Q_i^{(h)}$ is a box of side length $(N\ell - 1, 0)$ and $\mathcal{C}_i^{(v)} \subset Q_j^{(v)}$ is a box with side lengths $(\lfloor\theta_B^{5/4}\rfloor - 1, N\ell - 1)$. We call the grid *good* if $W_{\mathbf{j}}$ is B-traversable for any $\mathbf{j} \in \cup_i\mathcal{C}_i^{(h)}$ and $\mathcal{C}_j^{(v)}$ is a vertical crossing for each $j \in [N-1]$.

We have that $|\mathcal{C}_i^{(h)} \cap \mathcal{C}_j^{(v)}| = O(\theta_B^{5/4})$ and that on a grid we require this part to be B-traversable, AC-traversable and to contain an AC-super box at the same time. This is well-defined since AC-super states are a subset of AC-traversable states which in turn are subsets of B-traversable states.

For a grid $\mathcal{C}$ let $X(\mathcal{C})$ be the vertices given by $\mathbf{j}_{i,j} = \partial^{(r)}\mathcal{C}_i^{(v)} \cap \mathcal{C}_j^{(h)}$ for $i, j \in [N-1]$, where we recall that $\partial^{(r)}$ is the right boundary. We define the event $\mathcal{E}^{(1)}$ as the $\omega \in \Omega^*$ such that there is a good grid $\mathcal{C}$ and there are $i, j \in [N-1]$ with $i, j > N/2$ such that $W_{\mathbf{j}_{i,j}}$ is B-super.

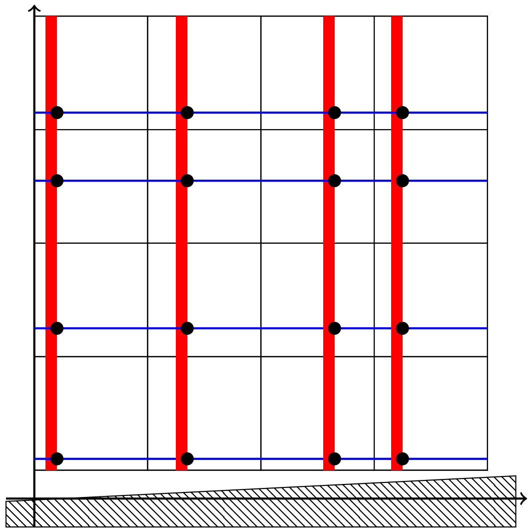

FIGURE 9.12 Grid $\mathcal{C}$ as defined in Definition 9.36. The horizontal configurations from Lemma 9.35 are in blue, the vertical crossings from Definition 9.31 in red, a bit thicker to represent the horizontal extension of $\theta_B^{5/4}$. The black circles form the set $X(\mathcal{C})$. The striped area indicates the area on which we cannot condition by the exterior condition theorem together with the exhausting family of sets $\{V_n\}_{n\in\mathbb{Z}}$ defined above Definition 9.31.

Lemma 9.37. *For any $\varepsilon > 0$ we find a $q(\varepsilon)$ such that*

$$(N\ell)^2\mu(1-\mathbb{1}_{\mathcal{E}^{(1)}}) \leq \varepsilon$$

if $q_B < q(\varepsilon)$.

Proof. We take the union bound of $\mu(1-\mathbb{1}_{\mathcal{E}^{(1)}})$ of the event of not finding all the horizontal paths, vertical crossings or the B-super intersection point. Not finding all the horizontal paths is upper bounded with the union bound of there being one $Q_i^{(h)}$ such that each row contains a B on the outer vertices, i.e. upper bounded by

$$CN(1-(1-q_B)^{\kappa N\ell})^{\ell} \leq CN(\kappa q_B N\ell)^{\ell} \leq 2^{-O(\theta_B^{5/2})}$$

for q_B small enough. Multiplied by $(N\ell)^2$ this is still decreasing in q_B. For the vertical crossings we look at the $\ell/\lfloor\theta_B^{5/4}\rfloor = O(\theta_B^{1/4})$ disjoint boxes in each strip that could be vertical crossings and upper bound the failure probability to find any vertical crossing by the probability that none of these disjoint boxes are vertical crossings. The probability of not being a vertical crossing is upper bounded by either having B-vacancy outside the central vertex on the right boundary or if there is no AC-super state. Thus,

$$\left(1-(1-q_B)^{\kappa N\ell\theta_B^{5/4}} + (1-q_Aq_C(1-q_B))^{\kappa\theta_B^{5/4}}\right)^{N\ell\lfloor\theta_B^{1/4}\rfloor} \leq 2^{-\kappa\theta_B^{5/4}}$$

for q_B small enough, which still decreases after multiplying with $(N\ell)^2$. Finally, the probability of not finding a B-super $W_{\mathbf{j}_{i,j}}$ is upper bounded by

$$(1-q_B)^{(N/2)^2} \leq e^{-O(\theta_B^2)}$$

which again decreases even after multiplying with $(N\ell)$ and the claim follows. $\square$

Combining these events we can bring a B-super vertex to $\mathbf{j}_{0,0}$ for the respective good grid given by $\mathcal{E}^{(1)}$. As before, we need to bring the B-super box to a deterministic vertex. Since the grid this time starts at $\mathbf{e}_2$ we can immediately bring it back to the origin. Let $\mathcal{E}^{(2)}$ be the event that $\mathbf{W}_\mathbf{j}$ is

- AC-traversable for $\mathbf{j}$ either in $\Gamma^{(1)} := \{-\lfloor\theta_B^{5/4}\rfloor\mathbf{e}_1, \dots, -\mathbf{e}_1\}$ or $\Gamma^{(2)} := \Gamma^{(1)} + \mathbf{e}_2$ and there is at least one $\mathbf{j}$ in both $\Gamma^{(1)}$ and $\Gamma^{(2)}$ with $W_\mathbf{j}$ AC-super.
- B-traversable for $\mathbf{j}$ in $\Gamma^{(3)} := \{\mathbf{e}_2, \dots, (\ell-1)\mathbf{e}_2\}$ (i.e. the left boundary of $Q_{0,0}$).

In an analogous calculation to Lemma 9.37 we get.

Lemma 9.38. *For any $\varepsilon > 0$ we find a $q(\varepsilon)$ such that*

$$\ell\mu(1 - \mathbb{1}_{\mathcal{E}^{(2)}}) \leq \varepsilon$$

if $q_B < q(\varepsilon)$.

Let $\mathcal{E} := \mathcal{E}^{(1)} \cap \mathcal{E}^{(2)}$ and let $\mathcal{E}_x$ be the event translated by $x \in \mathbb{Z}^2$. $\{\mathcal{E}_x\}_x$ satisfies the exterior condition w.r.t. the $\{V_n\}_{n\in\mathbb{Z}}$ defined above and thus using Lemmas 9.37 and 9.38 we get that we can apply the exterior condition theorem, Theorem 8.2, with this family of events. We come to the proof of part (3.iii).

Proof of Theorem 6(3.iii). By the exterior condition theorem we have

$$\mathrm{Var}(f) \leq 4 \sum_{x\in\mathbb{Z}^2} \mu(\mathbb{1}_{\mathcal{E}_x}\mathrm{Var}_x(f)).$$

Let us upper bound the summand for $x = 0$. First use that $\mathrm{Supp}(\mathcal{E}_0) \cap W_\mathbf{0} = \emptyset$ to extend the variance (Lemma 9.15)

$$\mu(\mathbb{1}_{\mathcal{E}_0}\mathrm{Var}_0(f)) \leq \mu(\mathbb{1}_{\mathcal{E}_0}\mathrm{Var}^*_\mathbf{0}(f)).$$

For $\omega \in \mathcal{E}$ let $\mathcal{G}(\omega)$ denote the unique good grid in ω consisting of the lowest horizontal paths and vertical crossings in the $\prec$-order that make a good grid. Further let $\xi \in X(\mathcal{G})$ be the largest intersection point that is B-super in the lexicographic order. Let $\mathcal{E}_{\mathcal{C},\mathbf{j}_{i,j}}$ be the event $\mathcal{E}$ with $\mathcal{G} = \mathcal{C}$ and $\xi = \mathbf{j}_{i,j}$. We have

$$\mu(\mathbb{1}_{\mathcal{E}_0}\mathrm{Var}^*_\mathbf{0}(f)) = \sum_{\mathcal{C}\text{ grid}} \sum_{n,m\in[N]} \mu(\mathbb{1}_{\mathcal{E}_{\mathcal{C},\mathbf{j}_{n,m}}}\mathrm{Var}^*_\mathbf{0}(f)).$$

Further let $\mathcal{E}^{(i,j)}_{\mathcal{C},\mathbf{j}_{n,m}}$ for $(i,j) \in [0,n] \times [0,m-1]$ be the part of the event $\mathcal{E}_{\mathcal{C},\mathbf{j}_{n,m}}$ that depends on the vertices outside the i-th vertical strip and j-th horizontal strip, if $i > n$ or $j > m-1$ let $\mathcal{E}^{(i,j)}_{\mathcal{C},\mathbf{j}_{n,m}} = \mathcal{E}_{\mathcal{C},\mathbf{j}_{n,m}}$. We have

$$\sum_{\mathcal{C}\text{ grid}} \sum_{n,m\in[N]} \mathbb{1}_{\mathcal{E}^{(i,j)}_{\mathcal{C},\mathbf{j}_{n,m}}} \leq 2\ell$$

since only the grid outside of the $Q_i^{(h)}$ and $Q_j^{(v)}$ is fixed and inside these strips there are at most ℓ choices of straight horizontal paths or boxes that could be vertical crossings respectively (in the latter case ℓ is a rough estimate of $\ell/\lfloor\theta_B^{5/4}\rfloor$).

Fix a grid $\mathcal{C}$ and $n, m \in [N]$, extend the variance (Lemma 9.15) and use the block relaxation Lemma (Lemma 6.17) to get

$$\mu(\mathbb{1}_{\mathcal{E}_{\mathcal{C},\mathbf{j}_{n,m}}}\mathrm{Var}^*_\mathbf{0}(f)) \leq 2^{\kappa\theta_B}\mu\Big[(\mathbb{1}_{\mathcal{E}_{\mathcal{C},\mathbf{j}_{n,m}}}\Big(\mathbb{1}_{W_{\mathbf{j}_{0,0}}\ B\text{-super}}\mathrm{Var}^*_\mathbf{0}(f) + \mathrm{Var}^{(*,BT)}_{\mathbf{j}_{0,0}}(f)\Big)\Big]. \tag{9.10}$$

We extend the variance in the first summand to $\{0, \mathbf{e}_2\}$ and then use the block relaxation Lemma again:

$$\begin{aligned}&\mu(\mathbb{1}_{\mathcal{E}_{\mathcal{C},\mathbf{j}_{n,m}}, W_{\mathbf{j}_{0,0}}\ B\text{-super}}\mathrm{Var}^*_{\mathbf{0}}(f)))\\&\quad\le 2^{\kappa\theta_B}\mu\Big[\mathbb{1}_{\mathcal{E}_{\mathcal{C},\mathbf{j}_{n,m}}, W_{\mathbf{j}_{0,0}}\ B\text{-super}}(\mathbb{1}_{W_{\mathbf{e}_2}\ B\text{-super}}\mathrm{Var}^*_{\mathbf{0}}(f) + \mathrm{Var}^{(*,BT)}_{\mathbf{e}_2}(f))\Big].\end{aligned} \tag{9.11}$$

For the second summand in Equation (9.11) there is a unique shortest path Γ from $\mathbf{e}_2$ to $\mathbf{j}_{0,0}$ first on the bottom boundary of $D_{0,0}$ and then following the grid $\mathcal{C}$. Through a combination of extending the variance, the block relaxation Lemma, Lemmas 9.33 and 9.35 we get

$$\mu\Big[\mathbb{1}_{\mathcal{E}_{\mathcal{C},\mathbf{j}_{n,m}}, W_{\mathbf{j}_{0,0}}\ B\text{-super}}\mathrm{Var}^{(*,BT)}_{\mathbf{e}_2}(f)\Big] \le 2^{\kappa\theta_B^{3/2}}\mu\Big[\mathbb{1}_{\mathcal{E}^{(0,0)}_{\mathcal{C},\mathbf{j}_{n,m}}}\mathcal{D}_{\Gamma\cup\mathrm{Supp}(\mathcal{E}^{(2)})}(f)\Big].$$

Analogously for the first term in Equation (9.11) using Lemma 9.33. We can then take the sum over $\mathcal{C}$, n and m and absorb the overcounting of the vertices in $\mathrm{Supp}(\mathcal{E}^{(2)})$ into κ above for q_B small enough.

For the second summand in Equation (9.10) assume without loss of generality that $n = m = N$ and define the constraints $c^{(\mathcal{C})}_{\mathbf{j}_{i,j}}$ for $\mathbf{j}_{i,j} \in X(\mathcal{C})$ that there is a B-super state on $\mathbf{j}_{i+1,j} \in X(\mathcal{C})$ or on $\mathbf{j}_{i,j+1} \in X(\mathcal{C})$ and $c^{(\mathcal{C})}_{\mathbf{j}_{N-1,N}} \equiv 1$. The auxiliary dynamics on $\cup_{i\in[N-1],j\in[N]}\mathbf{j}_{i,j}$ with constraints $c^{(\mathcal{C})}_{\mathbf{j}_{i,j}}$ is equivalent to a two-dimensional East process with minimal boundary conditions on $(0,2)$-squeezed box. By Proposition 6.6(i) we find a subset $V \subset X(\mathcal{C})$ such that

$$\begin{aligned}\mu(\mathbb{1}_{\mathcal{E}_{\mathcal{C},\mathbf{j}_{N,N}}}\mathrm{Var}^{(*,BT)}_{\mathbf{j}_{0,0}}(f)) &\le \mu(\mathbb{1}_{\mathcal{E}_{\mathcal{C},\mathbf{j}_{N,N}}}\mathrm{Var}^{(*,BT)}_{V}(f))\\&\le 2^{\theta_B^2(1+\varepsilon)/4}\sum_{\mathbf{j}_{i,j}\in V}\mu(\mathbb{1}_{\mathcal{E}_{\mathcal{C},\mathbf{j}_{N,N}}}c^{(\mathcal{C})}_{\mathbf{j}_{i,j}}\mathrm{Var}^{(*,BT)}_{\mathbf{j}_{i,j}}(f))\\&\le 2^{\theta_B^2(1+\varepsilon)/4}\sum_{i\in[N-1],j\in[N]}\mu(\mathbb{1}_{\mathcal{E}_{\mathcal{C},\mathbf{j}_{N,N}}}c^{(\mathcal{C})}_{\mathbf{j}_{i,j}}\mathrm{Var}^{(*,BT)}_{\mathbf{j}_{i,j}}(f)).\end{aligned}$$

Now $\mathbb{1}_{\mathcal{E}_{\mathcal{C},\mathbf{j}_{N,N}}}c^{(\mathcal{C})}_{\mathbf{j}_{i,j}}$ is upper bounded by the constraints that either the top or right neighbour of $\mathbf{j}_{i,j}$ is B-super. Taking this upper bound we recover Dirichlet forms of the ABC-model analogously to the first summand in Equation (9.10) by using Lemmas 9.33 and 9.35 and finally summing $\mathbb{1}_{\mathcal{E}^{(i,j)}_{\mathcal{C},\mathbf{j}_{n,m}}}$. We end up with an estimate

$$\mathrm{Var}(f) \le 2^{\theta_B^2(1+\varepsilon)/4}\sum_{x\in\mathbb{Z}^2}\mu(\mathcal{D}_{\mathrm{Supp}(\mathcal{E}^{(2)}_x)\cup\mathcal{Q}^{(B)}_x}(f)).$$

Thus the overcounting of each term is of order $O(N\ell)$ which we can absorb into ε to get the claim. □

Chapter 10

Conclusions and open problems

In the first part of this thesis (Chapters 3 and 7) we proved bounds on the front evolution speed of the multidimensional East process in certain q-dependent directions in the limit of $q \to 0$ and a mixing result behind the front in $\mathbb{Z}^d$. In the second part (Chapters 4, 8 and 9) we introduced a process, the MCEM, which roughly can be described as multiple mutually exclusive East processes evolving on the same lattice. For this process we proved finiteness of the spectral gap in certain cases and for $d = 2$ showed that for a wide range of vacancy density configurations, the spectral gap is comparable to the spectral gap of a two-dimensional East process in the limit $q_{\min} \to 0$.

The technical backbone behind both results (Chapter 6) is the result that for the East process on $\mathbb{Z}^d$, even if starting from minimal boundary conditions, there is a set around the main diagonal which relaxes on the same time scale as the East process started from maximal boundary conditions. This agrees with the intuition that near the diagonal there are exponentially more oriented paths that can reach any one vertex compared to vertices near the axes.

These results leave a set of open questions that we were not be able to solve. We present these with short paragraphs on why our methods fail or alternatively potential routes to tackle the problems.

10.1 Front evolution problem

Conjecture 10.1. *For any direction $\mathbf{x} \in \mathbb{R}_+^d$ we have $v_{\max}(\mathbf{x}) = v_{\min}(\mathbf{x})$.*

Let us discuss where our proof methods fall short of proving this conjecture. Theorem 1(B) only bounds the minimal front speed from the front speed of the one-dimensional East process, but in fact our proof implies a lower bound on $v_{\min}$. Recall that in Equation (7.11) for the front velocity in directions slowly approaching the axis we found the bound

$$\limsup_{q\to 0} -\frac{1}{\theta_q^2}\log_2(v_{\min}(x)) \leq \frac{\phi(\beta;d)}{2},$$

and then used Proposition 6.6(ii) saying that $\phi(\beta;d) < 1$ for $\beta \in (0,1)$. If we go back to the proof of Proposition 6.6(ii) we see that this resulted from the induction in d giving $\phi(\beta;d) \leq (\phi(\beta;d-1) \vee \phi(\beta;2)) < 1$ and Equation (6.8) that says that

$$\phi(\beta;2) \leq \frac{1}{2}(1-\beta)^2 + 2\beta - \beta^2.$$

Thus, included in our results already is a slightly stronger result saying that

$$\limsup_{q\to 0} -\frac{1}{\theta_q^2}\log_2(v_{\min}(x)) \leq \frac{1}{4}(1-\beta)^2 + \beta - \frac{1}{2}\beta^2.$$

This is a bound that goes from $1/4$ when $\beta = 0$ to $1/2$ when $\beta = 1$ i.e. from a one-dimensional to a two-dimensional lower bound for $v_{\min}$.

If we want to get a d-dimensional bound when $\beta = 1$ instead of doing an induction in d we need to make an explicit d-dimensional construction, i.e. a d-dimensional version of Figure 6.4. To do this, the partition of the (β, κ)-squeezed box into a coarse grained $(0, 2)$-squeezed box is analogous. Then instead of the construction in Figure 6.4 we could have taken the *Knight chain and repeat the same steps we took for (i), leading to a bound

$$\phi(\beta; d) \leq \frac{1}{d}(1-\beta)^2 + 2\beta - \beta^2.$$

Given such a lower bound on $v_{\min}(x)$ as above we would like to find the fitting upper bound for $v_{\max}(x)$, but using our techniques we are still far from that goal. Indeed in $d = 2$, the upper bound given in Theorem 1(C) on $v_{\max}$ goes from the one-dimensional to the two-dimensional bound for angles that scale with θ_q^2, i.e. if $\max_{i,j} \mathbf{x}_i(q)/\mathbf{x}_j(q) \geq 2^{\alpha\theta_q^2}$, the one-dimensional bound is given for $\alpha = 1/4$ and the two-dimensional for $\alpha = 0$. The lower bound on $v_{\min}$ on the other hand scales linearly in θ_q, i.e. if $\max_{i,j} \mathbf{x}_i(q)/\mathbf{x}_j(q) \leq 2^{2\alpha\theta_q}$, and again gives the one-dimensional bound for $\alpha = 1/4$ and the two-dimensional for $\alpha = 0$.

Thus, there is a large gap between the bounds. In the bulk or close to the axis they are comparable but for the transition from the bulk to the axes the question is still wide open. Furthermore, the bounds that we provide do not suggest whether one of the two bounds is closer to reality.

Related to this is the following question:

Question 10.2. *If Conjecture 10.1 holds, can we identify the function of the velocity $v(\mathbf{x}) = v_{\max}(\mathbf{x}) = v_{\min}(\mathbf{x})$ depending on $\mathbf{x}$.*

Indeed, Conjecture 10.1 does not necessarily imply an answer to this question as we saw in the one-dimensional case. We recall that in [7] Blondel proved the existence of a front by proving that the process seen from the front had a stationary measure. So if it is possible the generalise the proof in [7] to $\mathbb{Z}^d$ or otherwise prove the conjecture the question of the exact scaling of $v(\mathbf{x})$ still remains. As mentioned, above our results do not give a hint to which scaling is the more accurate one.

Recall that we define

$$C(t) = \{x \in \mathbb{R}_+^d : \tau_x \leq t\}.$$

Answering these two questions above then leads to the conjecture already stated in Section 3.1

Conjecture 10.3. *There exists a compact subset $\hat{C} \subset \mathbb{R}_+^d$ such that*

$$\forall \varepsilon > 0 \quad \lim_{t\to\infty} \mathbb{P}_{\omega^*}\Big((1-\epsilon)t\hat{C} \subseteq C(t) \subseteq (1+\epsilon)t\hat{C}\Big) = 1.$$

Generalising Theorem 1(C) to d-dimensions In Remark 3.7 we said that Theorem 1(C) could have been presented in d-dimensions as well with a similar proof. Indeed, it seems that a construction of a U_x analogous to Figure 7.1 in which the top-right part is a hyperplane of dimension at most $d-1$, chosen correspondingly depending on how $\mathbf{x}$ approaches the axes, i.e. whether it approaches just one axis or multiple at the same time. The proof and statement are analogous albeit more involved, but in light of the above mentioned open questions it seems that the more pressing open problem is to bring $v_{\max}$ and $v_{\min}$ closer in $d = 2$ before aiming for $d > 2$ constructions.

Relating to the cutoff result, we only give cutoff in a specific case that is governed by the one-dimensional mode. Thus the next question suggests itself.

Question 10.4. *Is there cutoff for the East process on $\mathbb{Z}^d$ with maximal boundary conditions on Λ_n or for other geometries than Λ_n.*

Our proof of Theorem 4 crucially relied on the fact that, given minimal boundary conditions, the modes along the axes are much slower than in any other direction so we could use the one-dimensional cutoff result, Theorem 3.5, to get cutoff on Λ_n. When we have different boundary conditions on Λ_n or we consider other geometries like $\Lambda(\delta, \varepsilon, t)$ for $\delta > 0$ from Theorem 3 this obviously does not work anymore and at the very least we would probably need an answer to Conjecture 10.1.

10.2 MCEM process

Conjecture 10.5. *The G-MCEM process is ergodic for any $|G| \leq 2^d - 1$.*

Theorem 5 shows that the G-MCEM for $G = H_d$ is not ergodic and then shows positivity of the spectral gap, and thus in particular ergodicity, for G such that all $h \in G$ share a propagation direction (case (B.i)), in which case $|G| \leq 2^{d-1}$, or that G (or a superset thereof) is a star graph (case (B.ii)), in which case $|G| \leq d + 1$. An immediate question that arises from this is whether the chain for $G = 2^d - 1$ is ergodic or not. In the case of $d = 2$ the answer is yes since any G of cardinality $2^d - 1 = 3$ is a star graph, but for $d \geq 3$ we have no result.

It is clear that the proof for the case (B.i) cannot be used for G where all vacancy types share a direction. Let us outline the proof only of ergodicity in the star graph case to showcase how a generalisation to G with $|G| = 2^d - 1$ might look.

Lemma 8.8 is sufficient to show the ergodicity of the model where G is a star graph. Indeed, consider the event $\mathcal{A}_x(n, m)$ comprised of the configurations for which $\Lambda + x - n\mathbf{v}$ is good and that in the box $\Lambda_x(n, m)$, of side length $n + m$ that contains x and of which $x - n\mathbf{v}$ is a corner, every interval of length m contains every vacancy type. Then, it is not hard to see that $\mu(\cup_{n,m}\mathcal{A}_x(n, m)) = 1$ for any x and that given a $\mathcal{A}_x(n, m)$ and using Lemma 8.8 repeatedly, it is possible to find a legal path of length depending on n, m and d that allows to put any vacancy type on x. Instead of using the exterior condition theorem, which is not applicable since $\mathcal{A}(n, m)$ does not satisfy the exterior condition, we can then use

$$\mathrm{Var}(f) \leq \sum_x \mu(\mathrm{Var}_x(f)) = \sum_{x,n,m} \mu(\mathbb{1}_{\mathcal{A}_x(n,m)}\mathrm{Var}_x(f))$$

and the path method to upper bound any summand in the right hand side by a term proportional to $\mathcal{D}_{\Lambda_x(n,m)}(f)$. Assuming that $\mathcal{D}(f) = 0$ implies that $\mathcal{D}_{\Lambda_x(n,m)}(f) = 0$ and thus $\mathrm{Var}(f) = 0$ which in turn implies that f is constant and thus the chain associated to the G-MCEM is ergodic by Theorem 2.2.

The question is thus how the proof methods of Lemma 8.8 can be generalised to apply to G with $|G| = 2^d - 1$. If we take the analogous definition of good Λ for equilateral boxes Λ of side length 2 as the configuration ω such that $\omega_{2h} = h$ for each $h \in G$ and $\omega_x = \star$ for all $x \in \cup_i F_i \setminus G$, where we note that $G \subset \cup_i F_i$.

The first part of the proof of Lemma 8.8, relaxing $\Lambda \setminus \cup_i F_i$, then applies equally to this case with $2^d - 1$ vacancy types. The problem is with making $\Lambda + \mathbf{v}$ good since on the paths $\{\mathbf{v} + j\mathbf{e}_i : j \in [k_i - 1]\}$, we were looking for the only vacancy type such that $\mathbf{e}_i \notin \mathcal{P}(h)$, i.e. h_i, but if $G = 2^{d-1}$ then there are up to $2^{d-1} - 1$ such vacancy types we could meet. Further, $2h + \mathbf{v}$ does not immediately neighbour $\Lambda \setminus \cup_i F_i$ anymore which we used before to argue that we can clear the paths in the $\mathbf{e}_i$ direction of any non-h_i-vacancy.

So it is not clear that defining good Λ in this way can be used to prove ergodicity for $|G| = 2^d - 1$

Question 10.6. *In the ABC-model are there parameter sets $\mathbf{q}$ with $q_{\min} = q_C$ and $\liminf_{q_C \to 0} q_{\text{med}} > 0$ such that*

$$\lim_{q_C \to 0} \frac{\gamma(G, \mathbf{q})}{\gamma_2(q_C)} = 1?$$

Theorem 6 does not treat the case of having many A- and B-vacancies in the ABC-model. This is because their propagation directions are exactly opposite. It was crucial that A and C vacancies share a direction in the proof of Theorem 6(3.iii). Since they shared a direction it sufficed to require that there be a single AC-super box in every row, since this excludes A- and C-vertices on the top and bottom vertices respectively this is not an event we can require for every vertex in a row and in fact it sufficed that we require that the rest of each row contains no B-vacancy to allow two-dimensional relaxation of the B-vacancies.

In the case of many A- and B-vacancies we have not managed to find a configuration that facilitates two-dimensional motion of C-vacancies, but it seems likely that we can prove one-dimensional motion, i.e. prove that there for any ε there is a $q_C(\varepsilon)$ such that

$$\gamma(G, \mathbf{q}) \geq \gamma_1(q_C)^{(1+o(1))}$$

for all $\mathbf{q}$ with $q_C < q_C(\varepsilon)$ and $\liminf_{q_C \to 0} q_{\text{med}} > 0$. To see this, recall the proof of Theorem 5(B.ii) and in particular Figure 8.4. In that case we considered the ACD-model. Notice that the good event $\mathcal{E}^{(N)}$ only required one A-vacancy (the central vacancy, i.e. the analogue of the C-vacancy in the ABC-model), while it required $O(N)$ C- and D-vacancies. Thus, if $q_C, q_D > \lambda$ for some $\lambda > 0$ it seems probable that we can use the path method at a constant cost to remove any C or D vacancies in the way and then move the A-vacancy one-dimensionally to the origin.

If we can affirmatively answer Question 10.6 the next conjecture is the logical next step.

Conjecture 10.7. *Fix $\Delta > 0$ and consider a G-MCEM on $\mathbb{Z}^2$ with $|G| \leq 3$ and let $\mathbf{q}$ be a valid parameter set with $p > \Delta$. Then,*

$$\lim_{q_{\min} \to 0} \frac{\gamma(G, \mathbf{q})}{\gamma_2(q_{\min})} = 1.$$

Proving this does not only require an answer to Question 10.6, but also a more refined construction in the cases (3.i) through (3.iii) since the conditions we require on $\mathbf{q}$ for these cases leave some gaps. Consider for example the case where $q_{\max}\theta_{q_{\min}}^{10} \to 0$ as $q_{\min} \to 0$. Morally, we would still say that there is no frequent vacancy type, but Theorem 6 makes no statement about this case. Thus, a more refined construction for the various cases would be necessary to prove Conjecture 10.7.

This would solve the two-dimensional case and begs the question: how does this generalise to d-dimensions.

Question 10.8. *What conditions can we put on G and $\mathbf{q}$ such that*

$$\lim_{q_{\min} \to 0} \frac{\gamma(G, \mathbf{q})}{\gamma_d(q_{\min})} = 1?$$

Related to this is, of course, Conjecture 10.5. At first glance it seems the methods from Chapter 9 generalise to d-dimensions once a d-dimensional version of the h-grid $\mathcal{Q}^{(h)}$ is constructed. Consider $(3.i)$ in which the salient point is that we find paths that contain only one vacancy type at most, which generalises to the d-dimensional case where each vacancy type has a low density. The analogue of (3.ii) is the case where at most one vacancy type is frequent. In this case, it seems again like the ideas from the two-dimensional case should be recyclable since we can always require there to be boxes with the frequent vacancy type on boundaries of the boxes as we did in Section 9.3. Now for any higher number of frequent vacancy types, it might be possible to recycle the construction from (3.iii) which needed mainly, as discussed above, that the frequent vacancy types share a direction but this needs some more careful thought.

Recall that the initial inspiration for the MCEM was the model given in [27], in which additionally to the East mechanics it was also possible that for example a B-vacancy facilitated by an A-vacancy could flip back to the neutral state. We called this ring on a vertex a *diffusive ring* and there was a rate parameter $\xi \in [0, 1]$ choosing between the classic MCEM and the diffusive ring (see Remark 4.7). We called this the isotropic MCEM and the parameter $\xi \in [0, 1]$ was the rate of diffusive rings.

Question 10.9. *What is the scaling of the spectral gap of the isotropic MCEM as* $q_{\min} \to 0$ *for* $\xi \in (0, 1)$.

In [27] Chandler and Garrahan conjecture, based on simulations, that the resulting spectral gap of this model is an interpolation with parameter ξ between the spectral gap of FA-1f and that of the multidimensional East model. In fact, it seems natural to conjecture that for $\xi > 0$ at least ergodicity should not be a problem anymore, even though $G = H_d$, since there is a non-zero chance that previously blocking vacancy types can remove each other. The author is not aware of any research in this direction currently ongoing making this an open problem.

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